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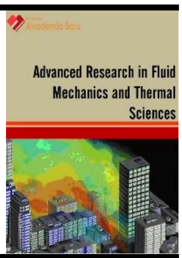

Instantaneous Thermal-Diffusion and Diffusion-Thermo Effects on Carreau Nanofluid Flow Over a Stretching Porous Sheet

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ABSTRACT

In this study, the system of nonlinear partial differential equations of boundary-layer steady flow of Carreau nanofluid over a moving stretching sheet is modulated. Moreover, both thermal diffusion and diffusion thermo effects are considered. This system has been simplified into a system of nonlinear ordinary differential equations using appropriate similarity transformations. Then it has been solved by using the multi-step differential transformation method and the differential transformation method with Padé approximation. These methods represent approximations with a high degree of accuracy and minimal computational effort for studying the particle motion in a steady boundary layer flow and heat transfer over a porous moving plate in presence of thermal radiation. In addition, the velocity, temperature and nanoparticles concentration profiles are obtained and depicted graphically in the current study. The porosity parameter effect on the stretching velocity is analyzed and it is shown that the increase of porosity parameter tends to reduce the stretching velocity.

Keywords:
Boundary-layer flow; Heat generation; Thermal radiation; Padé-differential transformation method approximation; Multi-step differential transform method

1. Introduction

In fluid mechanics, the boundary layer is considered as an essential part and indicates to the layer of fluid in the immediate vicinity of a bounding surface where the effects of viscosity are considerable. A range of velocities occurs across the boundary layer from maximum to zero, provided that the fluid is in contact with the surface. The expansion of boundary layer velocity on a flat plate

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was first scrutinized by Blasius [1] and that's expansion of velocity have been studied through this paper. In recent years. Considerable attention has been devoted to the study of boundary layer flow behavior and heat transfer characteristics of a non-Newtonian fluid past a vertical plate because of its extensive applications in engineering processes. Howarth [2] discussed the various aspects of the Blasius flat plate flow problem numerically while, our paper use two different semi-analytical methods. The boundary layer equations play a central role in many aspects of fluid mechanics since they design the motion of a viscous fluid near the surface. The heat transfer for problem that offered by Howarth [2] was computed by Pohlhausen [3]. Abou-zeid [4] studied the heat generation and viscous dissipation effects on a Newtonian fluid over a stretching sheet with heat transfer. El-dabe *et al.,* [5] discussed effects of uniform magnetic field, heat generation and chemical reaction on the flow of non-Newtonian nanofluid down a vertical cylinder. For more theorems and investigation see Refs. [6-10].

In many transport processes in nature, flow is driven by density differences caused by temperature gradient, chemical composition (concentration) gradient and material composition. It is therefore important to study flow induced by concentration differences independently or simultaneously with temperature differences. The energy flux caused by the composition gradient is called Dufour effect (diffusion-thermo). If the mass fluxes are created by temperature gradients, it is called Soret effect (thermal-diffusion). Abou-zeid [11] computed the influences of thermal-diffusion and viscous dissipation on peristaltic flow of micropolar non-Newtonian nanofluid using homotopy perturbation method. These effects are generally of a small order of magnitude. The Dufour effect was found to be of an order of magnitude such that it cannot be neglected [12]. Soret effect plays an important role in the operation of solar ponds, biological systems, and the microstructure of the world oceans. In biological systems mass transport across biological membranes induced by small thermal gradients in living matter is an important factor. Soret effect also is utilized for isotope separation and, in a mixture of gases of light molecular (H2, He) and medium molecular weight (N2, air). Eldabe *et al.,* [13] studied non-Darcian, radiation and chemical reaction effects on MHD non-Newtonian nanofluid flow over a stretching sheet Through a porous medium. Osalusi *et al.,* [14] examined numerically the effects of thermal-diffusion and diffusion thermo on combined heat and mass transfer of a steady hydromagnetic convective slip flow due to a rotating disk with viscous dissipation and Ohmic heating. Recently, Srinivas *et al.,* [15] (several references therein) studied the thermal diffusion and diffusion thermo effects in a two-dimensional viscous flow between slowly expanding or contracting walls with weak permeability. For more details see Refs. [16-20].

The main aim of this paper is to make comparisons between two new algorithms Ms-DTM and Pade'-DTM to solutions of thermal diffusion and diffusion thermo effects on the flow of Carreau nanofluid over a moving stretching sheet with variable thickness. A computer program technique is presented by using Mathematica 11 to discuss and cure the slow convergent rate or completely divergent in the wider region that happened in DTM techniques with aid of the multi-step differential (Ms-DTM) transform method and pretend the Pade'-DTM techniques. Influences of pertinent physical parameters on velocity, temperature and concentration have been discussed. To approve the validity of our results and our solutions the semi-numerical and analytical results that obtained by Ms-DTM and Pade'-DTM are compared with the nearest published results by Khan *et al.,* [21].

2. Mathematical Formulations

In this paper, we consider a mathematical model to study Carreau nanofluid flow due to nonlinear stretching sheet with variable thickness. The sheet is stretched with velocity $U_w = U_0(x+b)^m$ where U_0 reference velocity. Suppose thickness of sheet is $y = B(x+b)^{\frac{1-m}{2}}$ (see Figure 1). Here b is the dimensionless constant and m is the power law index. We observe that the model must be satisfied only for $m \neq 1$, because for $m = 1$, the problem reduces to flat sheet. Suppose that the variable magnetic field of strength $B_o(x)$ is applied vertical to the plate.

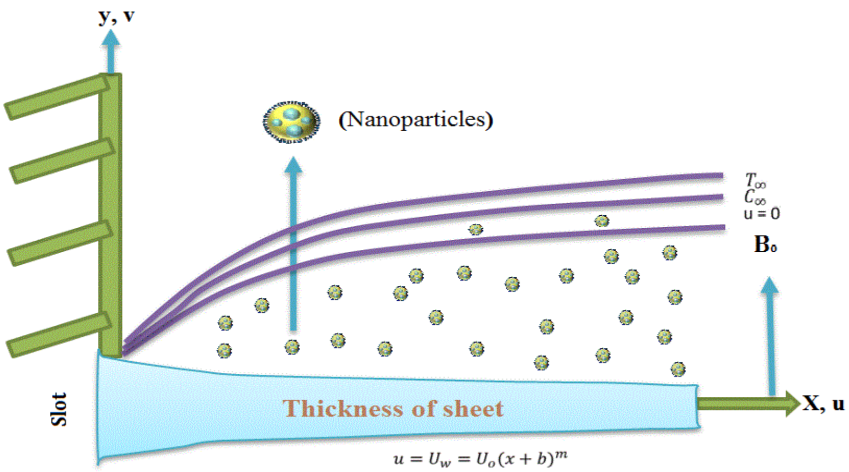

Fig. 1. Geometry of the problem

In view of all above approximations and by applying the boundary layer approximation, the governing equation of motion can be written as [22-24]:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \left(v\frac{\partial^2 u}{\partial y^2} + v\frac{3(n-1)}{2}\Gamma^2\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}\right) - \frac{v}{k}u - \frac{\sigma B_0^2 u}{\rho} \tag{2}$$

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \alpha\frac{\partial^2 T}{\partial y^2} + \tau\left\{D_B\left(\frac{\partial T}{\partial y}\frac{\partial C}{\partial y}\right) + \frac{D_T}{T_\infty}\left[\left(\frac{\partial T}{\partial y}\right)^2 + \left(\frac{\partial T}{\partial x}\right)^2\right]\right\} + \frac{D_m k_T}{C_s}\frac{\partial^2 C}{\partial y^2} - \frac{\partial q_r}{\partial y} + Q_0(T - T_\infty), \tag{3}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D_B\left(\frac{\partial^2 C}{\partial y^2}\right) + \frac{D_T}{T_\infty}\left(\frac{\partial^2 T}{\partial y^2}\right) + \frac{D_m k_T}{T_m}\frac{\partial^2 T}{\partial y^2} - k_r(C - C_\infty) \tag{4}$$

With associated boundary conditions,

$$u\left(x + B(x+b)^{\frac{1-m}{2}}\right) = U_0(x+b)^m, v\left(x + B(x+b)^{\frac{1-m}{2}}\right) = 0, T\left(x + B(x+b)^{\frac{1-m}{2}}\right) = T_w, \tag{5}$$

$$u(x,\infty) = 0, T \to T_\infty, C \to C_\infty, \text{as } y \to \infty. \tag{6}$$

By using self-similarity transformations,

$$\eta = y\sqrt{U_0\left(\frac{m+1}{2}\right)\left(\frac{(x+b)^{m-1}}{v}\right)},$$
$$\Psi = \sqrt{vU_0\left(\frac{2}{m+1}\right)(x+b)^{m-1}f(\eta)},\ \Theta(\eta) = \frac{T-T_\infty}{T_w-T_\infty}, \phi(\eta) = \frac{C-C_\infty}{C_w-C_\infty}, \tag{7}$$

where the stream function Ψ is defined in usually way as $u = \frac{\partial \Psi}{\partial y}$ and $v = -\frac{\partial \Psi}{\partial x}$.

Eq. (1)-(4) with boundary conditions Eq. (5) becomes

$$f''' + \frac{2m}{m+1}(f')^2 - ff'' + \frac{3(n-1)}{2}{W_e}^2(f'')^2 f''' - \left(\frac{1}{K} + M\right) f' = 0, \tag{8}$$

$$\left(\frac{1+R}{P_r}\right)\theta'' + \theta' f + N_b \phi' \theta' + N_t(\theta')^2 + \gamma\theta + D_u\theta'' = 0, \tag{9}$$

$$\phi'' + S_c L_e P_r\, f\, \phi' + \left(S_r S_c + \frac{N_t}{N_b}\right)\theta'' - S_c L_e \lambda\, \phi = 0, \tag{10}$$

The corresponding transformed boundary conditions are,

$$f(\eta) = 0, f'(\eta) = 1, \theta(\eta) = 1,\ \phi(\eta) = 1, \eta \to 0, f(\infty) = 0, \theta(\infty) = 0, \phi(\infty) = 0, \tag{11}$$

where $M = \frac{2\sigma B_0^2}{(m+1)\rho U_0 (x+b)^{m-1}}$, $W_e = \frac{U_0^3 (x+b)^{3m-1}}{2v}$, $P_r = \frac{\mu c_p}{k}$, $\gamma = \frac{2Q_0 (x+b)^{m-1}}{(1+m)\rho c_p U_0}$, $N_b = \frac{D_B (C_w - C_\infty)}{v}$, $R = \frac{4\sigma^* T_\infty^3}{k_\infty k^*}$, $\lambda = \frac{2k}{(m+1) U_0 (x+b)^{m-1}}$, $L_e = \frac{v}{D_B}$, $N_t = \frac{D_t (T_w - T_\infty)}{v}$, $S_r = \frac{D_m k_T}{v T_m}\left(\frac{T_w - T_\infty}{C_w - C_\infty}\right)$, $D_u = \frac{D_m k_T}{v C_s C_p}\left(\frac{C_w - C_\infty}{T_w - T_\infty}\right)$, $K = \frac{k U_0}{v x}$ and $S_c = \frac{v}{D_m}$.

Physically quantity of primary interest (i.e.) the rate of heat transfer is given by,

$$NuRe_x^{-1/2} = -\sqrt{\frac{(m+1)}{2}}\theta'(0),$$

where $Re_x^{1/2} = \frac{U_0 (x+b)}{v}$.

Eq. (8)-(10) with associated boundary conditions Eq. (11) are solved semi-numerically by using multi-step differential transform techniques and Pade'-DTM algorithm. Comparisons have been made between those mentioned methods through graphs and tables.

3. Methodology

In this attempt, firstly, solutions are offered using differential transform method (DTM) which introduced for the first time by Zhou [25]. The differential transform technique is one of the semi-numerical analytical methods for ordinary and partial differential equations that use the form of polynomials as approximations of the exact solutions that are sufficiently differentiable [26–29]. For convenience of the reader, we present a review of the DTM.

Consider a general equation of nth order ordinary differential equation,

$$y(t, f, f', \ldots, f^{(n)}) = 0. \tag{12}$$

Subject to the initial equations,

$$f^{(k)}(0) = d_k,\ k = 0, \ldots, n-1. \tag{13}$$

Let $f(t)$ be analytic in a domain D and let $t = t_0$ represent any point in D. The function $f(t)$ is then represented by one power series whose centre is located at t_0. The differential transformation of the k^{th} derivative of a function $f(t)$ is defined as the following:

$$F(k) = \left(\frac{1}{k!}\right)\left[\left(\frac{d^{(k)}f(t)}{dt^{(k)}}\right)\right]_{(t=t^0)},\ \forall t \in D. \tag{14}$$

And the inverse transformation of $F(k)$ can take the form,

$$f(t) = \sum_{k=0}^{\infty} F(k)(t-t_0)^{(k)},\ \forall t \in D. \tag{15}$$

In fact, from Eq. (14) and Eq. (15), we obtain,

$$f(t) = \sum_{k=0}^{\infty} \frac{(t-t_0)^{(k)}}{k!}\left(\frac{d^{(k)}y(t)}{dt^{(k)}}\right)_{t=t_0}, \forall t \in D. \tag{16}$$

Eq. (15) implies that the concept of differential transformation is derived from the Taylor series expansion. In real applications, the function $f(t)$ is expressed by a finite series and Eq. (16) can be written as:

$$f(t) = \sum_{k=0}^{N} F(k)(t-t_0)^{(k)},\ \forall t \in D. \tag{17}$$

Eq. (17) implies that $\sum_{k=N+1}^{\infty} F(k)(t-t_0)^{(k)}$ is negligibly small.

The following table show that the transformation for some functions and relation by using differential transformation method. In view of the differential transform method and the operations of differential transformation given in Table 1, Eq. (8)-(10) with associated boundary conditions Eq. (11) are,

$$[k+1][k+2][k+3]F[k+3] + \frac{2m}{m+1}\sum_{r=0}^{k}(r+1)(k-r+1)F(r+1)F(k-r+1) - \sum_{r=0}^{k}(r)(k-r+1)(k-r+2)F(r)F(k-r+2) + \frac{3(n-1)}{2}{W_e}^2\sum_{r_2}^{k}\sum_{r_1=0}^{r_2}(r_1+1)(r_2-r_1+1)(k-r_2+1)(k-r_2+2)(k-r_2+3)\,F(r_1+1)F(r_2-r_1+1)F(k-r_2+2)F(k-r_2+3) - \left(\frac{1}{k}+M\right)[k+1]F[k+1] = 0 \tag{18}$$

$$\left(\frac{1+R}{P_r}\right)[k+1][k+2]\Theta[k+2] + \sum_{r=0}^{k}(r)(k-r+1)F(r)\Theta(k-r+1) + \sum_{r=0}^{k}N_b(r+1)(k-r+1)\Phi(k-r+1)\Theta(k-r+1) + N_t\sum_{r=0}^{k}(r+1)(k-r+1)\Theta(r+1)\Theta(k-r+1) + \gamma\Theta[k] + D_u[k+1][k+2]\Theta[k+2] = 0 \tag{19}$$

$$[k+1][k+2]\Phi[k+2] + S_cL_eP_r\sum_{r=0}^{k}(r)(k-r+1)F(r)\Phi(k-r+1) + \left(S_rS_c + \frac{N_t}{N_b}\right)[k+1][k+2]\Theta[k+2] - S_cL_e\lambda\,\Phi[k] = 0, \tag{20}$$

where $F[k], \Theta[\mathrm{k}]$and $\Phi[k]$ are the differential transformations of $f[\eta], \theta[\eta]$and $\phi[\eta]$ respectively. The differential transform of the boundary conditions are given by,

$$F[0]=0,\sum_{k=0}^{n}F(k)h^k=1,\Theta[0]=1,\Phi[0]=1,\sum_{k=0}^{\infty}F(k)h^k=0,\sum_{k=0}^{\infty}\Theta(k)h^k=0\sum_{k=0}^{\infty}\Phi(k)h^k=0, \quad (21)$$

On applying the DTM technique, the given differential equations and their related boundary / initial conditions are transformed into recurrence a relation who leads to the solution of a system of algebraic equations as coefficients of a power series solution. In some situations where the governing equations of the system contain highly non-linear terms, the solutions may diverge. To overcome the shortcoming, the Pade'-DTM and Ms-DTM are applied.

3.1 Pade'-DTM Approximation

Padé approximant is the best approximation of a function by a rational function of given order – under this technique, the approximant's power series agrees with the power series of the function it is approximating. The technique was developed around 1890 by Henri Padé, but goes back to Georg Frobenius who introduced the idea and investigated the features of rational approximations of power series. The Padé approximant often gives better approximation of the function than truncating its Taylor series and it may still work where the Taylor series does not converge. For these reasons Padé Padé-DTM technique which is the combination between Pade' approximation and differential transform method has been adopted and used recently by Zheng [30] to overcome such a difficulty in DTM results and to get approximate solutions for a wide range. In literature review, Representative studies dealing with Pade'-DTM techniques can be found in [31-34]. Solution obtained by DTM is in terms of power series. Since the radius of convergence of the power series may not be large enough to contain the two boundaries, Padé approximants are applied to manipulate the obtained series for numerical approximations to overcome this difficulty. Padé approximant is the best approximation for a polynomial approximation of a function into rational functions of polynomials of given order. Let the power series $\sum_{i=0}^{\infty}a_i x^i$, represent a function $f(x)$; i.e.,

$$f(x)=\sum_{i=0}^{\infty}a_i x^i. \quad (22)$$

The Padé approximant is a rational function given by,

$$[L/M]=\frac{P_L(x)}{Q_M(x)} \quad (23)$$

where $P_L(x)$ is a polynomial of degree atmost L and $Q_M(x)$ is a polynomial of degree at most M. And we have:

$$f(x)=a_0+a_1x+a_2x^2+a_3. \quad (24)$$

$$P_L(x)=p_0+p_1x+p_2x^2+p_3x^3+\cdots+p_Lx^L. \quad (25)$$

$$Q_M(x)=q_0+q_1x+q_2x^2+q_3x^3+\cdots+q_Mx^M. \quad (26)$$

It may be observed that in Eq. (25), there are $L+1$ numerator and $M+1$ denominator coefficients. Since we can clearly multiply numerator and denominator by a constant and leave $[L/M]$ unchanged, we impose the normalization condition.

$$Q_M(0) = 1. \tag{27}$$

Thus there are $L + 1$ independent numerator and M independent denominator coefficients, making $L + M + 1$ unknown coefficient in all. This number suggests that normally $[L/M]$ ought to fit the power series Eq. (22) through the orders.

$$1, x, x^2, x^3, \ldots, x^{1+M}. \tag{28}$$

So, in the notation of formal power series, we have:

$$\sum_{i=0}^{\infty} a_i x^i = \frac{p_0+p_1x+p_2x^2+p_3x^3+\cdots+p_Lx^L}{q_0+q_1x+q_2x^2+q_3x^3+\cdots+q_Mx^M} + O(x^{L+M+1}). \tag{29}$$

By cross multiplying, we get that:

$$(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots)(q_0 + q_1x + q_2x^2 + q_3x^3 + \cdots + q_Mx^M) = p_0\, p_1x + p_2x^2 + p_3x^3 + \cdots + p_Lx^L + O(x^{L+M+1}). \tag{30}$$

Equating the coefficients of $1, x, x^2, x^3, \ldots, x^L$ and the coefficients of $x^{L+1}, x^{L+2}, \ldots, x^{L+M}$, we get the following sets of equations:

$$\begin{cases} a_0 = p_{0,} \\ a_1 + a_0q_1 = p_{1,} \\ a_2 + a_1q_1 + a_0q_2 = p_{2,} \\ \quad\vdots \\ a_L + a_{L-1}q_1 + a_{L-2}q_2 + \cdots + a_0q_L = p_{L,} \end{cases} \tag{31}$$

and,

$$\begin{cases} a_{L+1} + a_Lq_1 + \cdots + a_{L-M+1}q_M = 0, \\ a_{L+2} + a_{L+1}q_1 + \cdots + a_{L-M+2}q_M = 0, \\ a_{L+3} + a_{L+2}q_1 + \cdots + a_{L-M+3}q_M = 0, \\ \quad\vdots \\ a_{L+M} + a_{L-M+1}q_1 + \cdots + a_Lq_M = 0, \end{cases} \tag{32}$$

If $n < 0$, we take $a_n = 0$ for consistency and $q_j = 0$ for $j > M$. On solving Eq. (31) and Eq. (32), we obtain:

$$[L/M] = \frac{\begin{vmatrix} a_{L-M+1} & a_{L-M+1} & \cdots & a_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ \sum_{j=M}^{L} a_{j-M}x^j & \sum_{j=M-1}^{L} a_{j-M+1}x^j & \cdots & \sum_{j=0}^{L} a_jx^j \end{vmatrix}}{\begin{vmatrix} a_{L-M+1} & a_{L-M+1} & \cdots & a_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ x^M & x^{M-1} & \cdots & 1 \end{vmatrix}} \tag{33}$$

The Eq. (31) and Eq. (32) normally determine the Padé numerator and denominator and are called Padé equations. The $[L/M]$ Padé approximant is constructed in Eq. (33).

In addition, new algorithm named by Padé-DTM technique which is the combination between Pade' approximation and differential transform method has been adopted and used recently by Zheng [30] to overcome such a difficulty in DTM results and to get approximate solutions for a wide range. The solutions are obtained by using Pade'-DTM are extracted as the following steps:

Firstly, the series solution of velocity profile are extracted from the obtained results by DTM

$$f = Sum[F[i]t^\wedge i, \{i, 0, 10\}]/.Q0/.a[[1]]$$
$$t - 1.60409\, t^3 + 1.66924\, t^3 - 1.36504\, t^4 + 0.877565\, t^5 - 0.484399\, t^6 + 0.196149\, t^7 - 0.0148953\, t^8 - 0.0987711\, t^9 + 0.152201\, t^{10}.$$

Then, we apply the Laplace transformation on the obtained series and find

$$LaplaceTransform[f, t, s]$$
$$552305./s^{11} - 35842./s^{10} - 600.58/s^9 + 988.59/s^8 - 348.767/s^7 + 105.308/s^6 - 32.7609/s^5 + 10.0155/s^4 - 3.20817/s^3 + 1/s^2.$$

To simplify the obtained series by Laplace transform we replace each s by $\frac{1}{s}$ as:

$$LaplaceTransform[f, t, s]/.(s) -> 1/s$$
$$s^2 - 3.20817\, s^3 + 10.0155\, s^4 - 32.7609\, s^5 + 105.308\, s^6 - 348.767\, s^7 + 988.59\, s^8 - 600.58\, s^9 - 35842.\, s^{10} + 552305.\, s^{11}.$$

Now, we apply the Pade' approximation technique on the last step at the convenient order

$$PadeApproximant[g2, \{s, 0, \{5,4\}\}]$$
$$(1.000000000000\, s^2 + 14.3733\, s^3 + 17.7772\, s^5 - 234.836\, s^6)/(1.000000000000 + 17.5815\, s + 64.1661\, s^2 - 172.306\, s^3 - 724.763\, s^4).$$

Then, we simplify the Pade' approximation and a part it series as the following form

$$g5 = Apart[g4]$$
$$0.324018/s - 0.00281611/(-3.1304 + 1.\, s) - 0.320238/(3.12978\ + 1.\, s) - 0.00101442/(6.9723\ + 1.\, s) + 0.0000500677/(10.6098\ + 1.\, s).$$

Finally, we use the inverse of Laplace Transformation to obtain the following series by technique which is named by Pde'-DTM,

$$f(\eta) = \eta - 1.5119186579828328\eta^2 + 1.6689557041988523\eta^3 - 1.2889995595025034\eta^4 + 0.8757527018964728\eta^5 - 0.4592545946185259\eta^6 + 0.1986115514774606\eta^7 - 0.017838632237848876\eta^8 - 0.0875100499607522\eta^9 + 0.1400479877794983\eta^{10}.$$

In addition we apply the same method for extracting the following solutions of the distributions to temperature and concentration:

$$\theta(\eta) = 1 - 0.5970052193316422\eta - 0.1246443420148\eta^2 + 0.07968855925387092\eta^3 - 0.03895310795120801\eta^4 + 0.01333513321730088 6\eta^5 + 0.005286651538654442\eta^6 - 0.009736388406569228\eta^7 + 0.008304896413733847\eta^8 - 0.004373202569979797\eta^9 + 0.0012433643698761 8\eta^{10}.$$

$$\phi(\eta) = 1 - 0.9477359031328704\eta + 0.5003238114793479\eta^2 - 0.318129518753323\eta^3 + 0.1542157005205127 8\eta^4 - 0.05146528287277169\eta^5 - 0.02284482827570353 4\eta^6 + 0.04020898301367976\eta^7 - 0.03398577161345650 6\eta^8 + 0.01784536380911336\eta^9 - 0.00505289869020735 75\eta^{10}.$$

3.2 Multi-step Differential Transform Method

The DTM is unworkable for solving partial differential equations with highly non-linear behavior at infinity, as in these cases the series solution does not exhibit the real behaviors of the problem but gives a good approximation to the true solution in a very small region and it has slow convergent rate or completely divergent in the wider region. For this meditate multi-step DTM [35] that has been deliberated for the analytical solution of the differential equations along the solicited domain. Multi-step DTM can be defined as the following [36-39].

Let $t \in [t_0, t_0 + L]$ be the domain of definition of the solution where subdivided into $m \in Z$ pieces, $L = mh$, here h is chosen sufficiently small so that the series converges in the subintervals $[t_i, t_{i+1}], t_i = t_0 + ih, \mathrm{i} = 1,2, \dots, \mathrm{m}$ of equal step size $h = \frac{L}{m}$ by using the nodes $t_i = ih$, with the initial values α_i, here α_i being approximately calculated by the sum of the Taylor series at the second boundary of the preceding subdomain, namely $\alpha_i = \sum_{k=0}^{n} [Y_k]_{i-1} h^k$. Following this procedure, then eventually one gets, step by step, an approximate solution in the whole domain. The procedure has also been utilized to evaluate boundary value problems. In that case, the initial value α_0 that corresponds to the first boundary condition is designed systematically via the parameters α_i, up to the second boundary $t_0 + L$ where the desired condition is dictated. Series solution for boundary layer flow system Eq. (18)–(20) can be obtained as,

$$f(\eta) = \sum_{k=0}^{N} F(k)\eta^n, \tag{34}$$

$$\theta(\eta) = \sum_{k=0}^{N} \Theta(k)\eta^n, \tag{35}$$

$$\phi(\eta) = \sum_{k=0}^{N} \Phi(k)\eta^n, \tag{36}$$

Now, according to the multi-step DTM, taking $K = n.m$, the series solution for boundary layer flow system Eq. (18)–(20) is given by,

$$f(\eta) = \begin{cases} \sum_{k=0}^{K} F_1(k)\eta^n, & \eta \in [0, \eta_1] \\ \sum_{k=0}^{N} F_2(k)(\eta - \eta_1)^n & \eta \in [0, \eta_1] \\ \vdots & \\ \sum_{k=0}^{N} F_m(k)(\eta - \eta_{m-1})^n & \eta \in [\eta_{m-1}, \eta_m] \end{cases} \tag{37}$$

$$\theta(\eta) = \begin{cases} \sum_{k=0}^{K} \Theta_1(k)\eta^n, & \eta \in [0, \eta_1] \\ \sum_{k=0}^{N} \Theta_2(k)(\eta - \eta_1)^n & \eta \in [0, \eta_1] \\ \vdots & \\ \sum_{k=0}^{N} \Theta_m(k)(\eta - \eta_{m-1})^n & \eta \in [\eta_{m-1}, \eta_m] \end{cases} \tag{38}$$

$$\phi(\eta) = \begin{cases} \sum_{k=0}^{K} \Phi_1(k)\eta^n, & \eta \in [0, \eta_1] \\ \sum_{k=0}^{N} \Phi_2(k)(\eta - \eta_1)^n & \eta \in [0, \eta_1] \\ \vdots & \\ \sum_{k=0}^{N} \Phi_m(k)(\eta - \eta_{m-1})^n & \eta \in [\eta_{m-1}, \eta_m] \end{cases} \tag{39}$$

Where F_i, Θ_i and Φ_i for $i = 1, 2, \dots, m$ satisfy recurrence relations Eq. (18)–(20).

The solutions of velocity, temperature and concentration distributions of system of Eq. (8)-(10) with associated boundary conditions Eq. (11) are offered as a series solutions using Pade'-DTM method as follows:

$$\begin{aligned} f(\eta) = \eta &- 1.5209901699307298\eta^2 + 1.6689832933496183\eta^3 - 1.2964835568595185\eta^4 \\ &+ 0.8759383685206605\eta^5 - 0.46172512311191494\eta^6 \\ &+ 0.19839205704814702\eta^7 - 0.01756184409040212\eta^8 \\ &- 0.08858392735421694\eta^9 + 0.1412265681046618\eta^{10}. \end{aligned}$$

$$\begin{aligned} \theta(\eta) = 1 &- 0.5970052193316422\eta - 0.1246443420148\eta^2 + 0.07968855925387092\eta^3 \\ &- 0.03932920100538327\eta^4 + 0.013220650335222814\eta^5 \\ &+ 0.005260554423356315\eta^6 - 0.009834252666459554\eta^7 \\ &+ 0.008342696606379948\eta^8 - 0.004397588440885738\eta^9 \\ &+ 0.0012435741404654501\eta^{10}. \end{aligned}$$

$$\begin{aligned} \phi(\eta) = 1 &- 0.9477359031328704\eta + 0.5003238114793479\eta^2 - 0.318129518753323\eta^3 \\ &+ 0.15571666916978166\eta^4 - 0.05100165474859062\eta^5 \\ &- 0.022747389505003282\eta^6 + 0.04060621561788177\eta^7 \\ &- 0.034140759418335775\eta^8 + 0.0179442462739804\eta^9 \\ &- 0.005053643736287518\eta^{10}. \end{aligned}$$

Table 1
Comparisons of velocity, temperature and concentration profile using Pade'-DTM and Ms-DTM techniques

η	$f(\eta)\ Pade' - DTM$	$f(\eta)\ Ms - DTM$	$Error$	$\theta(\eta)\ Pade - DTM$	$\theta(\eta)\ Ms - DTM$	$Error$	$\phi(\eta)\ Pade' - DTM$	$\phi(\eta)\ Ms - DTM$	$Error$
0.0	0	-2×10^{-16}	-2×10^{-16}	1	1	0	1	1	0
0.1	0.0857865	0.0857868	3×10^{-7}	0.941192	0.941192	-2×10^{-10}	0.901668	0.901668	8×10^{-10}
0.2	0.148429	0.148433	4×10^{-6}	0.880358	0.880358	3×10^{-8}	0.811483	0.811484	9×10^{-8}
0.3	0.1942	0.194216	0.00001	0.817833	0.817833	4×10^{-7}	0.728100	0.728102	1×10^{-6}
0.4	0.227662	0.2277043	0.00004	0.753887	0.753885	2×10^{-6}	0.650435	0.650444	8×10^{-6}
0.5	0.252142	0.2522270	0.00008	0.688741	0.68873	0.00001	0.577608	0.577643	0.00003
0.6	0.270068	0.2702173	0.00014	0.622573	0.622542	0.00004	0.508900	0.508900	0.00010
0.7	0.283215	0.2834563	0.00024	0.555533	0.555545	0.00007	0.443713	0.443960	0.00020
0.8	0.292883	0.2932519	0.00036	0.487747	0.487585	0.00016	0.381539	0.382006	0.00050
0.9	0.300029	0.3005697	0.00054	0.419014	0.419325	0.00031	0.321941	0.32295	0.00101
1.0	0.305357	0.3061286	0.00077	0.359812	0.350361	0.00054	0.264531	0.26322	0.00179
1.1	0.309395	0.3104718	0.00107	0.280945	0.280036	0.00090	0.208959	0.211931	0.00297
1.2	0.312539	0.3140205	0.00148	0.211156	0.20973	0.00140	0.154901	0.159577	0.00467
1.3	0.315102	0.3171144	0.00201	0.141070	0.138927	0.00214	0.102051	0.109096	0.00704
1.4	0.317334	0.3200445	0.00270	0.070760	0.067625	0.00310	0.050114	0.050348	0.010234
1.5	0.3194577	0.3230810	0.00362	0.000299	0.00406	0.00430	−0.001195	−0.013218	0.0144139

4. Results and Discussions

With the help of DTM-Padé and Ms-DTM techniques, we have found the analytical solutions of the transformed Eq. (8)-(10) under the boundary conditions Eq. (11). In addition, comparisons between those mentioned methods are computed and signified analytically through Table 1. To verify the accuracy of those techniques, we have compared our results as given in Table 2 with the corresponding results of Khan *et al.,* [21] in case of $M = S_c = S_r = D_u = 0$. All obtained results show and approve to the accuracy of our proposed techniques. Series of solutions are offered by using two compared method Pade'-DTM and Ms-DTM as follows with the same values of parameters.

Figure 2 and Figure 3 clarify the behaviour of both tangential velocity and temperature for different techniques, namely, DTM, Pade-DTM and Ms-DTM.

To have a better physical insight into the problem, the effects of Porosity parameter (k), the Hartmann number (M) on the Stretching velocity profile $f(\eta)$. For this purpose, Figure 4 and Figure 5 are plotted. It's obvious from Figure 4 and Figure 5 that the stretching velocity profile has a two opposite behavior under influences of k and M as in [12]. It can see that the velocity gradient reduces by increasing the values of Hartmann number M because the momentum boundary layer thickness enhances as M increases as in [25], but variation in Lorentz force reduces the velocity profile. Physically, the application of a transverse magnetic field on an electrically conducting fluid gives rise to a resistive-type of force called the Lorentz force. This force has the propensity to reduce speed for the motion of the fluid in the boundary layer. Figure 6, Figure 7 and Figure 8 displayr the effect of thermophoresis parameter (Nt), dimensionless Dufour number (D_u) and Porosity parameter (k) on temperature profile. It is seen from Figure 6 and Figure 7 that the effect of thermophoresis parameter (Nt) and Dufour number (D_u) are to increase temperature in the boundary layer as the heat flux is absorbed by the fluid which in turn increases the temperature of the fluid very close to the porous boundary layer and its effect diminishes far away from the boundary layer. Figure 8 elaborates the variation of Porosity parameter (k) on temperature profile. It is noticed that by enhancing the values of k the temperature distribution decreases. Physically, as the porosity increases, the temperature gradient steepens at the surface, accompanied by a decrease in the temperature in the outer boundary region. At a downstream location in the flow coming up from below, the outer region tends to have a temperature equal to the local ambient temperature. If the ambient temperature increase with height is rapid, the temperature in this region is unable to attain the local ambient temperature, even though the physical temperature does increase due to heat transfer mechanisms. Figure 9 shows the behavior of Porosity parameter (k) on the profile of concentration. It observed that the higher values of k increase the values of concentration profile. Figure 10 examines the variation on the profile of concentration for unlike values of heat generation parameter γ [21]. It is clear that the concentration rate decreases for higher values of heat source parameter γ. Concentration profile decreases when heat generation parameter γ is greater than zero, also opposite behavior is noticed for γ less than zero. Soret number (S_r) is considered increasing function in nanoparticle concentration distribution as shown in Figure 11. Furthermore, the increase in the fluid concentration brings about a commensurate increase in the level of collision of the fluid particles. And, this has the tendency of increasing the kinetic energy of the particles, which subsequently increases the velocity of fluid.

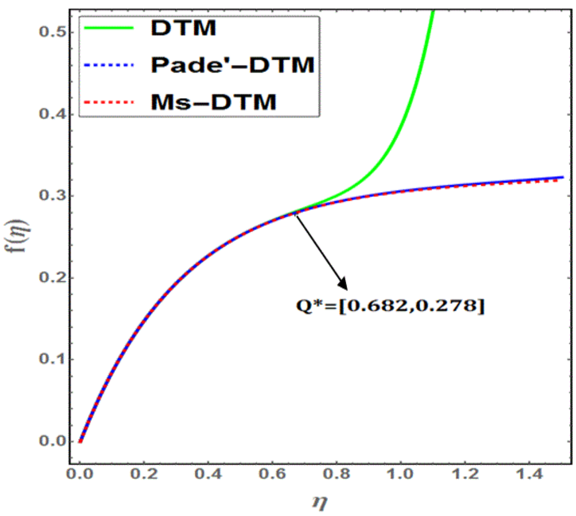

Fig. 2. Behavior of $f(\eta)$ solutions against η for different techniques

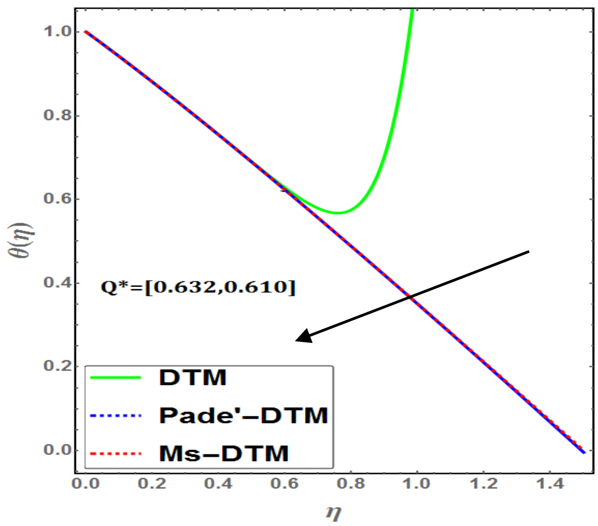

Fig. 3. Behavior of $\theta(\eta)$ solutions against η for different techniques

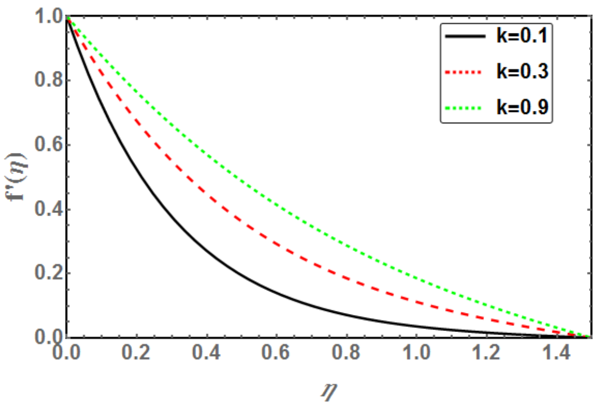

Fig. 4. Behavior of $f'(\eta)$ against η for several values k

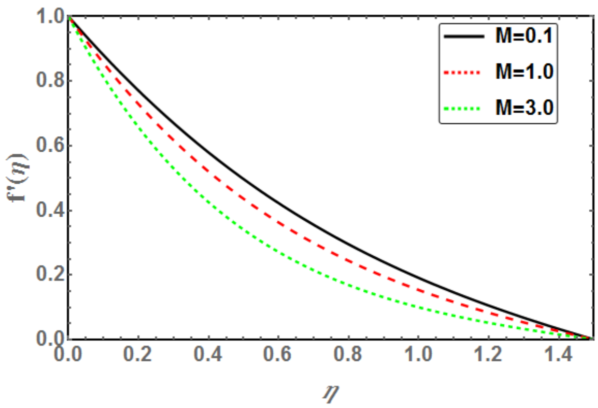

Fig. 5. Behavior of $f'(\eta)$ against η for several values M

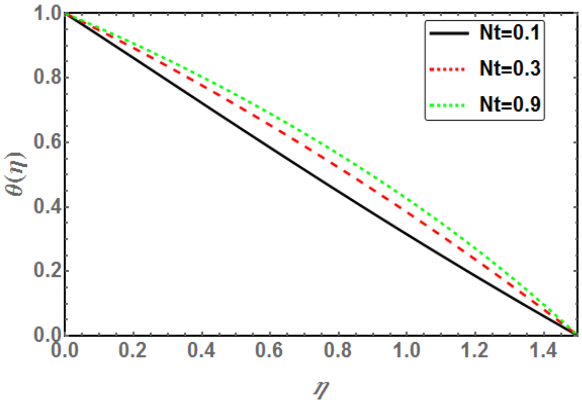

Fig. 6. Behavior of $\theta(\eta)$ against η for several values N_t

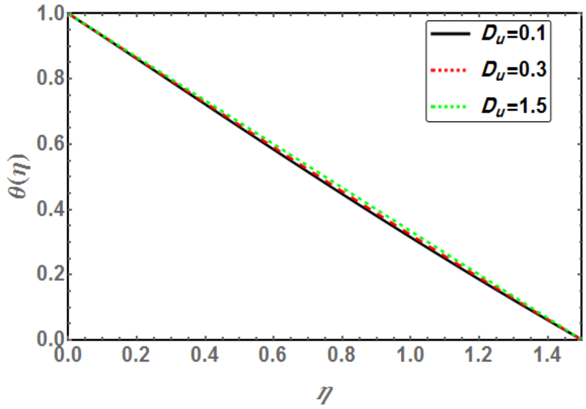

Fig. 7. Behavior of $\theta(\eta)$ against η for several values M

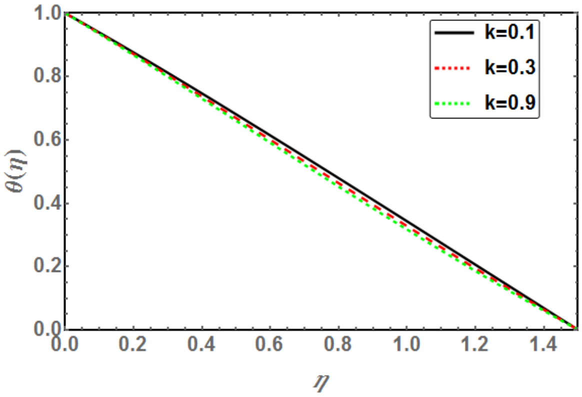

Fig. 8. Behavior of $\theta(\eta)$ against η for several values k

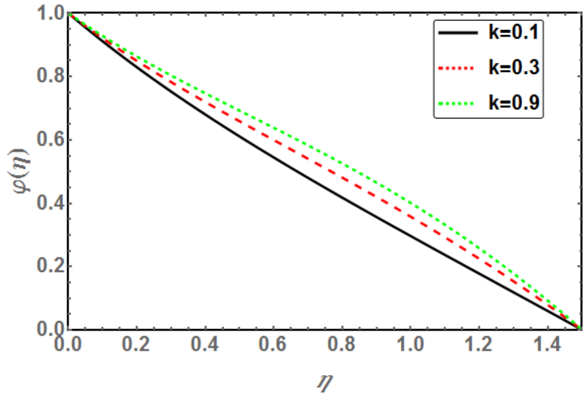

Fig. 9. Behavior of $\varphi(\eta)$ against η for several values M

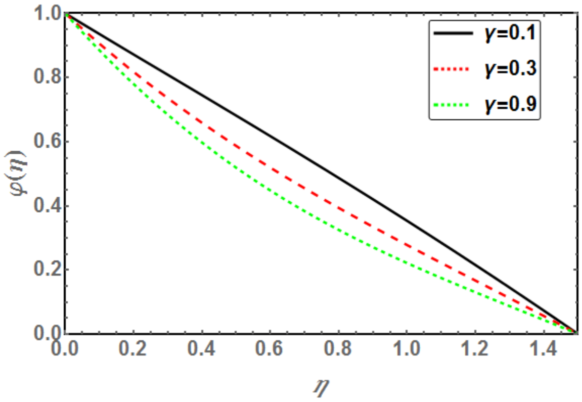

Fig. 10. Behavior of $\varphi(\eta)$ against η for several values M

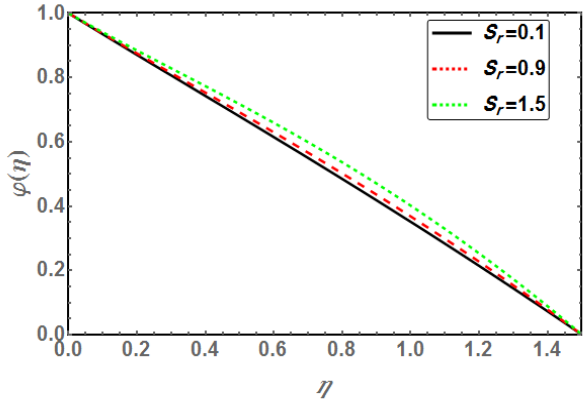

Fig. 11. Behavior of $\varphi(\eta)$ against η for several values S_r

Table 2
Comparison of $-\theta'(\eta)$ when $M = \alpha = \gamma = L_e = N_b = N_t = 0$ and $n = 1$

Pr	Khan *et al.* [21]	Present results by Pade'-DTM	Present results by Ms-DTM
0.07	0.0645	0.0649	0.0656
0.2	0.1663	0.1651	0.1661
0.7	0.4554	0.4539	0.4559
2.0	0.9100	0.9114	0.9100
7.0	1.8929	1.8905	1.8928
20.0	3.3505	3.3539	3.3504
70.0	6.4598	6.4522	6.4598

5. Conclusions

In this paper, we carefully present the Pade'-DTM and the multi-step DTM techniques, a reliable modifications of the DTM, which improves the convergence of the series solution. Those methods provide immediate and visible symbolic terms of solutions, as well as numerical approximate solutions to both linear and nonlinear differential equations. The validity of the proposed methods has been successful by applying it for boundary layer flow problem. The main outcomes of the present study are highlighted as:

i. Solutions which are obtained approve the validity of the two new algorithms.
ii. Pade'-DTM and Ms-DTM techniques doesn't need to any perturbation, linearization or restrictive assumptions to obtain the solutions related to highly nonlinear problems.
iii. DTM doesn't valid in a highly non-linear systems of equation.
iv. Porosity parameter and Hartmann number have an opposite effects on stretching velocity.
v. Heat generations and Porosity parameters have an opposite influences on concentration.

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