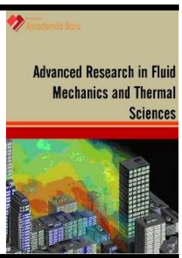
Journal of Advanced Research in Fluid Mechanics and Thermal Sciences 73, Issue 2 (2020) 106-125

Penerbit Akademia Baru

Journal of Advanced Research in Fluid Mechanics and Thermal Sciences

Journal homepage: www.akademiabaru.com/arfmts.html
ISSN: 2289-7879



Exact Equilibria for Ideal MHD Plasmas with Helically Symmetric Incompressible Flows and Variable Gravitational Field

Open Access

Salah Moawad[1,2,*], Omar El-Kalaawy[1], Heba Shaker[3]

[1] Department of Mathematics and Computer Science Department, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt
[2] Egyptian Korean Faculty of Technological Industry and Energy, Beni-Suef Technological University, Egypt
[3] Future High Institute of Engineering in Fyoum, Ministry of High Education, Fyoum, Egypt

ARTICLE INFO

Article history:
Received 24 January 2020
Received in revised form 2 May 2020
Accepted 5 May 2020
Available online 31 July 2020

Keywords:
Fluid dynamics; Helical symmetry; Incompressible flows; Exact equilibria

ABSTRACT

The equilibrium states of the system under consideration are governed by a nonlinear elliptic partial differential equation (PDE) for the helical magnetic flux function containing surface quantities along with a relation for the pressure. This leads to the study of equilibrium equations, which permit the derivation of several classes of analytic solutions with non-constant Mach number and differentially varying electric field perpendicular to the magnetic surfaces. The aim of this paper is the analytical solution of magnetohydrodynamic (MHD) flow. The equilibrium equations of motion for gravitating MHD plasmas are derived in the presence of incompressible mass flows with helical symmetry. The gravitational field is taken to be a variable vector function in a space of a cylindrical coordinates (r, ϕ, z). A similarity reduction approach is used to obtain exact solutions for several cases of the considered plasma flows with variable Mach number.

1. Introduction

A screw axis is a combination of a rotation and a translation along the axis of rotation in three-dimensional geometry. Within the context of ideal MHD theory, the equilibrium of helically symmetric plasmas with incompressible flows was investigated [1]. The motion of particles forming helical vortex filaments in a circular tube was considered in [2-5].

Most of the energy found in the matter that is beyond the black hole's gravitational force is ejected in jet form. Several models have been suggested for the formation of jets [6-9]. New evidence has emerged to support the theory that active galactic nuclei have helical magnetic fields, which could naturally collimate the jets. The properties of steady incompressible laminar flow of a

* *Corresponding author.*
E-mail address: salahmoawad@science.bsu.edu.eg

https://doi.org/10.37934/arfmts.73.2.106125

Newtonian fluid in helical pipes were studied in [10-13]. Completely designed flow was studied in a helical pipe with a view to modeling blood flow around the commonly non-planar bends in the arterial system [14].

The simplest mathematical model of fusion plasmas is provided by the set of MHD equations that provide a fluid explanation of the behavior of macroscopic plasma. Many researchers have achieved MHD equilibria with incompressible flows in cylindrical domains [15-25]. The mathematical complexity of the MHD equations has stood in the way of understanding hydromagnetic phenomena such as stellar winds [26-29], magnetic fields of solar prominence, and the confinement of laboratory plasma [30-31]. Several researchers [32-34] performed the derivation of the equilibrium equations of MHD plasmas in the literature. These are still expectedly general solutions for which few representative classes of MHD flows have been revised. The existence of stationary points for the dynamical system of Arnold-Beltrami-Childress flow was considered by Ershkov [35]. For this flow it has been shown that a three-parameter velocity field that provides a simple stationary solution of three dimensional Euler's equations for incompressible, inviscid fluids is the prototype for the study of turbulence.

Recently, non-stationary helical flows of the Navier-Stokes equations for incompressible fluids, with variable coefficient of proportionality between velocity and the curl field of flow, are investigated by Ershkov *et al.,* [36]. The case of constant proportionality was successfully investigated in Ershkov [37]. Some basic properties of helicity with reference to the constraining role of the magnetic helicity in the determination of stable magnetostatic structures are reviewed in [38]. A mathematical model for MHD slip Darcy boundary layer flow of viscoelastic fluid over a stretching surface with the presence of thermal radiation and viscous dissipation in a porous medium is formulated in [39].

Many helically symmetric MHD equilibria with incompressible flows are considered in the present work. The paper is structured as follows: in section 2, we discuss derivations of ideal incompressible MHD flows equations. Section 3 describes how to obtain the physical equilibrium variables of the examined MHD flows in section 2. In section 4, we derive the equation of motion of a gravitating plasma with incompressible flow and obtain exact solutions to the equilibrium equations for such flow. In section 5, we summarize the results.

2. Governing Equations for Helically Symmetric Incompressible Flows

In this section we consider stationary case of a quasi-neutral plasma with incompressible flow. In according to formulation in SI units convention, we assume the magnetic permeability of free space equals to 1. This MHD plasma is governed by the following set of equations. The incompressibility condition

$$\nabla \cdot \mathbf{v} = 0, \tag{1}$$

The momentum equation:

$$\rho(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla P + \mathbf{j} \wedge \mathbf{B}, \tag{2}$$

Faraday's law:

$$\nabla \wedge \mathbf{E} = 0, \tag{3}$$

Ampére’s law:

$$\nabla \wedge \mathbf{B} = \mathbf{j}, \tag{4}$$

The divergence-free Gauss law:

$$\nabla \cdot \mathbf{B} = 0, \tag{5}$$

Ohm’s law for MHD:

$$\mathbf{E} + \mathbf{v} \wedge \mathbf{B} = 0, \tag{6}$$

where $\rho, \mathbf{v}, P, \mathbf{B}$, $\mathbf{j}$ and $\mathbf{E}$ stand as usual for the mass density, flow velocity field, gas pressure, magnetic field (induction), vector of displacement current and electric field, respectively. The system under consideration is a magnetically confined plasma of a helically symmetric incompressible flow. To describe this configuration, we introduce cylindrical coordinates (r, ϕ, z), with z along the axis of rotation. Helical symmetry implies that any physical quantity depends only on r and u:

$$u = l\phi + kz, \tag{7}$$

where l and k are constant parameters characterize helical domain (with mandatory restriction to them of being not equal to zero).

The divergence-free fields, $\mathbf{B}$, $\mathbf{j}$ and $\rho\mathbf{v}$ can be expressed in terms of stream functions $\psi(r, u)$, $I(r, u)$, $F(r, u)$ and $\Theta(r, u)$ as

$$\mathbf{B} = I\mathbf{h} + \mathbf{h} \times \nabla\psi, \tag{8}$$

$$\mathbf{j} = (\pounds\psi - 2klh^2 I)\mathbf{h} - \mathbf{h} \times \nabla I, \tag{9}$$

and

$$\mathbf{v} = \frac{1}{\rho}(\Theta\mathbf{h} + \mathbf{h} \times \nabla F). \tag{10}$$

The electric field is expressed by

$$\mathbf{E} = -\nabla\Phi. \tag{11}$$

Using Eq. (8)-(11) and the vector $\mathbf{h}$ defined by

$$\mathbf{h} = \frac{l\nabla z - kr^2\nabla\phi}{l^2 + k^2 r^2}, \tag{12}$$

The MHD system in Eq. (1)-(6) is reduced to the following generalized Grad-Shafranov equation [1]:

$$(1-M^2)£\psi-\frac{1}{2}(M^2)'|\nabla\psi|^2-2klh^2X+\frac{1}{2}\left(\frac{X^2}{1-M^2}\right)'+\frac{1}{h^2}\left(p_s-\frac{XF'\Phi'}{1-M^2}\right)'+\frac{1}{2h^4}\left(\frac{\rho(\Phi')^2}{1-M^2}\right)'=0, \quad (13)$$

with $£\psi$, P, I and Θ determined as:

$$£\psi=\frac{1}{h^2}\nabla\cdot(h^2\nabla\psi)=\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\left(\frac{l^2-k^2r^2}{l^2+k^2r^2}\right)\frac{\partial}{\partial r}+\frac{l^2+k^2r^2}{r^2}\frac{\partial^2}{\partial u^2}\right]\psi, \quad (14)$$

$$P=p_s(\psi)-\rho\left(\frac{v^2}{2}+\frac{\Phi'\Theta}{\rho}\right), \quad (15)$$

$$I=\frac{X-F'\Phi'/h^2}{1-(F')^2/\rho}, \quad (16)$$

$$\Theta=IF'-\frac{\rho}{h^2}\Phi'=\frac{F'X-\rho\Phi'/h^2}{1-(F')^2/\rho}. \quad (17)$$

The dash denotes differentiation with respect to ψ. The symbol M denotes the Mach number which is defined by

$$M^2=\frac{v_h^2}{v_{Ah}^2}=\frac{(F')^2}{\rho}. \quad (18)$$

where v_h is the helical velocity and v_{Ah} is Alfvén velocity. For the above MHD flows, there are six surface quantities $F(\psi), \Phi(\psi), X(\psi), \rho(\psi)$, $p_s(\psi)$ and $M(\psi)$ five of them are arbitrary. These quantities are induced from the governing equations in Eq. (1)-(6) after using Eq. (7)-(12). They satisfy the relation $\mathbf{B}\cdot\nabla f(\psi)=0$ where the magnetic surfaces are defined by the relation $\psi=const$. Under the transformation [1, 18-19, 24]:

$$\Psi(\psi)=\int_0^{\psi}\sqrt{1-M^2(u)}\;du, \quad M^2<1, \quad (19)$$

Eq. (13) is reduced to

$$£\Psi-2klh^2\frac{X}{(1-M^2)^{\frac{1}{2}}}+\frac{1}{2}\frac{d}{d\Psi}\left(\frac{X^2}{1-M^2}\right)+\frac{1}{h^2}\frac{d}{d\Psi}\left(P_s-X\frac{dF}{d\Psi}\frac{d\Phi}{d\Psi}\right)+\frac{1}{2h^4}\frac{d}{d\Psi}\left[\rho\left(\frac{d\Phi}{d\Psi}\right)^2\right]=0 \quad (20)$$

For more details about the derivation of Eq. (20) [1].

3. Derivation of the Equations of Motion of A Gravitating Plasma Flow

The MHD equilibrium state of a gravitating plasma flow is governed by the set of Eq. (1), (3)-(6) in addition with the following momentum equation:

$$\rho(\mathbf{v}\cdot\nabla)\mathbf{v}=-\nabla P-\rho\nabla\Omega+\mathbf{j}\wedge\mathbf{B}, \quad (21)$$

where Ω is the gravitational potential.

Using Eq. (8)- (10), (16) and (17), the component of Eq. (21) along $\mathbf{B}$ and along $\nabla\psi$ are put in the consecutively forms:

$$\mathbf{B}\cdot\left(\nabla(\frac{v^2}{2}+\frac{\theta}{\rho}\Phi')+\frac{\nabla P}{\rho}+\nabla\Omega\right)=0, \tag{22}$$

and

$$\left\{\nabla\cdot\left[\left(1-\frac{(F')^2}{\rho}\right)h^2\nabla\psi\right]+\frac{F'F''}{\rho}h^2|\nabla\psi|^2-2klh^4X\right\}|\nabla\psi|^2+\left\{\frac{\rho}{2}\left[\nabla v^2-h^2\nabla\left(\frac{\theta}{\rho}\right)^2\right]+h^2\frac{\nabla I^2}{2}+\nabla P+\rho\nabla\Omega\right\}\cdot\nabla\psi=0. \tag{23}$$

Using $\rho=\rho(\psi)$, Eq. (22) can be integrated to yield an expression for the pressure as:

$$P=p_s(\psi)-\rho\left(\frac{v^2}{2}+\frac{\Phi'\Theta}{\rho}+\Omega\right), \tag{24}$$

where $p_s(\psi)$ is the static part of the pressure. In accordance with Eq. (15), we notice appearance of the gravitational potential at the right hand side. Eq. (24) is a generalization of Bernoulli invariant for steady flow of incompressible fluid.

Using Eq. (14), (16)-(19) and (24) into Eq. (23), we obtain the following elliptic PDE:

$$(1-M^2)£\psi-\frac{1}{2}(M^2)'|\nabla\psi|^2-2klh^2X+\frac{1}{2}\left(\frac{X^2}{1-M^2}\right)'+\frac{1}{h^2}\left(p_s-\frac{XF'\Phi'}{1-M^2}\right)'-\frac{1}{h^2}\Omega\rho'+\frac{1}{2h^4}\left(\frac{\rho(\Phi')^2}{1-M^2}\right)'=0. \tag{25}$$

Using the transformation (19), Eq. (25) is reduced to

$$£\Psi-2klh^2\frac{X}{(1-M^2)^{\frac{1}{2}}}+\frac{1}{2}\frac{d}{d\Psi}\left(\frac{X^2}{1-M^2}\right)+\frac{1}{h^2}\frac{d}{d\Psi}\left(P_s-X\frac{dF}{d\Psi}\frac{d\Phi}{d\Psi}\right)-\frac{1}{h^2}\Omega\frac{d\rho}{d\Psi}+\frac{1}{2h^4}\frac{d}{d\Psi}\left(\rho\left(\frac{d\Phi}{d\Psi}\right)^2\right)=0. \tag{26}$$

In the next sections, we show how to obtain the associated physical quantities to the full MHD system in Eq. (1)-(6). Moreover, we obtain several classes of exact solutions to Eq. (26) for some choices of the Mach number M.

4. Construction for Determination of The Equilibrium Physical Variables

In this section we explain how to obtain the equilibrium physical variables, $\mathbf{B},\mathbf{j},\mathbf{E},\sigma,P$ and ρ for the MHD system in Eq. (1)-(6) given in section 2. Form Eq. (19) we have

$$Q\equiv\frac{d\psi}{d\Psi}=\frac{1}{\sqrt{1-M^2}}. \tag{27}$$

Hence,

$$|\nabla\psi|^2=Q^2|\nabla\Psi|^2, \tag{28}$$

$$£\psi = Q£\Psi + \frac{dQ}{d\Psi}|\nabla\Psi|^2. \tag{29}$$

To express the physical quantities in terms of the new variable Ψ we introduce the following new vector:

$$\bar{\mathbf{B}} = \mathbf{h} \times \nabla\Psi. \tag{30}$$

Using Eq. (27)-(30) in Eq. (8)-(11) and (24), the equilibrium physical variables $\mathbf{v}, \mathbf{B}, \mathbf{j}, P$ can be determined as

$$\mathbf{v} = \frac{1}{\rho}\left(\Theta\mathbf{h} + \frac{dF}{d\Psi}\bar{\mathbf{B}}\right) = \frac{1}{Q}\left(\frac{\mathbf{B}}{\rho}\frac{dF}{d\Psi} - \frac{1}{h^2}\frac{d\Phi}{d\Psi}\mathbf{h}\right), \tag{31}$$

$$\mathbf{B} = Q\bar{\mathbf{B}} + I\mathbf{h}, \tag{32}$$

$$\mathbf{j} = \left[\left(Q£\Psi + \frac{dQ}{d\Psi}|\nabla\Psi|^2\right) - 2klh^2 I\right]\mathbf{h} - \mathbf{h} \wedge \nabla I = \left[\left(Q£\Psi + \frac{dQ}{d\Psi}|\nabla\Psi|^2\right) - 2klh^2 I\right]\mathbf{h} - \frac{dI}{d\Psi}\bar{\mathbf{B}} \tag{33}$$

$$\mathbf{E} = -\frac{d\Phi}{d\Psi}\nabla\Psi, \tag{34}$$

$$P = p_s(\Psi) - \rho\Omega - \rho\left[\frac{v^2}{2} + \frac{\Theta}{Q\rho}\left(\frac{d\Phi}{d\Psi}\right)\right] == p_s(\Psi) - \rho\Omega - \frac{1}{Q}\left[\frac{1}{2Q}\left(\frac{B^2}{\rho}\left(\frac{dF}{d\Psi}\right)^2 + \frac{\rho}{h^2}\left(\frac{d\Phi}{d\Psi}\right)^2\right) + \theta\frac{d\Phi}{d\Psi}\right]. \tag{35}$$

To obtain the solution $\Psi(r,u)$ of the original equation in Eq. (13), the Mach number should be chosen as a function of ψ (see Table 1).

Table 1
Some choices for the Mach number appeared in Eq. (18) and corresponding quantities ψ, Q and dQ/dΨ as functions of Ψ

No	M	ψ(Ψ)	Q(Ψ)	dQ/dΨ
1	$\varepsilon \equiv \text{const} < 1$	$\frac{1}{\sqrt{1-\varepsilon^2}}\Psi$	$\frac{1}{\sqrt{1-\varepsilon^2}}$	0
2	$\sqrt{\psi}$	$1-(1-\frac{3}{2}\Psi)^{2/3},\ 0 \le \Psi \le \frac{4}{3},\ \Psi \ne \frac{2}{3}$	$\frac{\sqrt[3]{2}}{(2-3\Psi)^{1/3}}$	$\frac{\sqrt[3]{2}}{(2-3\Psi)^{4/3}}$
3	$\sin(\psi)$	$\arcsin(\Psi),\ \Psi^2 < 1$	$\frac{1}{\sqrt{1-\Psi^2}}$	$\frac{\Psi}{(1-\Psi^2)^{3/2}}$
4	$\cos(\psi)$	$\arccos(1-\Psi),\ 0 < \Psi < 2$	$\frac{1}{\sqrt{2\Psi-\Psi^2}}$	$\frac{\Psi-1}{(2\Psi-\Psi^2)^{3/2}}$
5	$\tanh(\psi)$	$\ln[\tan(\Psi/2)],\ \Psi > 0$	$\sec(\Psi)$	$\sec(\Psi)\tan(\Psi)$
6	$\text{sech}(\psi)$	$\text{arccosh}(e^{\Psi}),\ \Psi > 0$	$\frac{e^{\Psi}}{\sqrt{e^{2\Psi}-1}}$	$-\frac{e^{\Psi}}{(e^{2\Psi}-1)^{3/2}}$

5. Exact Solution Classes for Eq. (26) and (21)

Case 1. Put

$$-2lk\frac{X}{\sqrt{1-M^2}} = A_0,\ \frac{d}{d\Psi}\left(P_s - X\frac{dF}{d\Psi}\frac{d\Phi}{d\Psi}\right) = A_1\Psi^2,\ \frac{d\rho}{d\Psi} = A_2\Psi^2,\ \frac{1}{2}\frac{d}{d\Psi}\left(\rho\left(\frac{d\Phi}{d\Psi}\right)^2\right) = A_3\Psi^2 \tag{36}$$

in Eq. (26) we get the PDE:

$$\frac{\partial^2\Psi}{\partial r^2}+\frac{1}{r}\left(\frac{l^2-k^2r^2}{l^2+k^2r^2}\right)\frac{\partial\Psi}{\partial r}+\left(\frac{l^2+k^2r^2}{r^2}\right)\frac{\partial^2\Psi}{\partial u^2}+\left(\frac{1}{l^2+k^2r^2}\right)A_0+ \\ ((l^2+k^2r^2)(A_1-A_2\Omega)+(l^2+k^2r^2)^2A_3)\Psi^2=0. \tag{37}$$

where $A_0, A_1, A_2 \neq 0$ and A_3 are constants. We restrict A_2 of being not equal to zero, according Eq. (43) mentioned below.

To seek exact solutions for Eq. (37), we consider the following new variable

$$\Psi(r,u)=\alpha(r,u)+\frac{u^2}{2}\beta(r,u). \tag{38}$$

Substituting Eq. (38) into Eq. (37) and equating the coefficients of like powers of u we get

$$u^4\!:\frac{\beta^2}{4}((l^2+k^2r^2)(A_1-A_2\Omega)+(l^2+k^2r^2)^2A_3)=0, \tag{39}$$

$$u^2\!:\alpha\beta[(l^2+k^2r^2)(A_1-A_2\Omega)+(l^2+k^2r^2)^2A_3]+\frac{(l^2+k^2r^2)}{2r^2}\beta_{uu}+\frac{1}{2r}\left(\frac{l^2-k^2r^2}{l^2+k^2r^2}\right)\beta_r \\ +\frac{1}{2}\beta_{rr}=0, \tag{40}$$

$$u^1\!:\frac{(l^2+k^2r^2)}{r^2}\beta_u=0, \tag{41}$$

$$u^0\!:\frac{A_0}{l^2+k^2r^2}+\alpha^2[(l^2+k^2r^2)(A_1-A_2\Omega)+(l^2+k^2r^2)^2A_3]+\frac{(l^2+k^2r^2)}{r^2}\beta+ \\ \frac{(l^2+k^2r^2)}{r^2}\alpha_{uu}+\frac{1}{r}\left(\frac{l^2-k^2r^2}{l^2+k^2r^2}\right)\alpha_r+\alpha_{rr}=0. \tag{42}$$

Eq. (39) gives

$$\Omega=\frac{1}{A_2}[A_1+(l^2+k^2r^2)A_3]. \tag{43}$$

Substitution of Eq. (41) and (43) into Eq. (40) yields

$$\beta(r)=c_2+c_1\left(\frac{k^2r^2}{2}+l^2\ln(r)\right). \tag{44}$$

Put

$$\beta(r)=-\alpha_{uu}, \tag{45}$$

in Eq. (44) and solve Eq. (42) and (45), we get:

$$\alpha(r,u)=\frac{1}{4}[4b_2+2b_1k^2r^2+u(4a_2-2c_2u+k^2r^2(2a_1-c_1u))+ \\ \frac{2}{k^2}(A_0+k^2l^2(2b_1+u(2a_1-c_1u)))\ln(r)], \tag{46}$$

where c_1, c_2, a_1, a_2, b_1 and b_2 are integration constants.

Thus, from Eq. (46) we get a solution for Eq. (26) as:

$$\Psi(r,u)=\frac{1}{4}[4b_2+2b_1k^2r^2+u(4a_2-2c_2u+k^2r^2(2a_1-c_1u))+\frac{2}{k^2}(A_0+k^2l^2(2b_1+u(2a_1-c_1u)))\ln(r)]+\frac{u^2}{2}\left[c_2+c_1\left(\frac{k^2r^2}{2}+l^2\ln(r)\right)\right]. \tag{47}$$

Using Eq. (7), solution (47) takes the form:

$$\Psi(r,\phi,z)=\frac{1}{4}[4b_2+2b_1k^2r^2+(l\phi+kz)(4a_2-2c_2(l\phi+kz)+k^2r^2(2a_1-c_1(l\phi+kz)))+\frac{2}{k^2}(A_0+k^2l^2(2b_1+2a_1(l\phi+kz)))\ln(r)]+\frac{(l\phi+kz)^2}{2}\left[c_2+c_1\left(\frac{k^2r^2}{2}\right)\right]. \tag{48}$$

Using the solution in Eq. (48), the quantity $\overline{\mathbf{B}}$ is obtained as:

$$\overline{\mathbf{B}}=-[\frac{a_2(l^2+k^2r)}{l^2+k^2r^2}+\frac{1}{2}a_1k^2r^2(l^3+k^2r)+4a_1l^2(l^2+k^2r)]\mathbf{e_r}+[\frac{1}{4}(4b_1k^2r+2k^2r(l\Phi+kz)(2a_1-c_1(l\Phi+kz))+\frac{1}{2}c_1k^2r(l\Phi+kz)^2+\frac{2l^2}{r}(2b_1+2a_1(l\Phi+kz))][\frac{-l\mathbf{e_\phi}+kr\mathbf{e_z}}{l^2+k^2r^2}]. \tag{49}$$

The solution of Eq. (21) can be obtained using Table 1. If we choose $M=\sin(\psi)$ as shown in Table 1, then $\psi_1=\arcsin(\Psi_1)$. Hence we obtain an exact solution to Eq. (21) as follows:

$$\psi_1=\arcsin[\frac{1}{4}[4b_2+2b_1k^2r^2+(l\phi+kz)(4a_2-2c_2(l\phi+kz)+k^2r^2(2a_1c_1(l\phi+kz)))+\frac{2}{k^2}(A_0+k^2l^2(2b_1+2a_1(l\phi+kz)))\ln(r)]+\frac{(l\phi+kz)^2}{2}[c_2+c_1(\frac{k^2r^2}{2})]]. \tag{50}$$

Using Eq. (31)-(35), we obtain the magnetic field, velocity field, electric field, vector of displacement current and gas pressure as:

$$\mathbf{B}=\frac{\overline{\mathbf{B}}}{\sqrt{1-\Psi_1^2}}+\frac{I}{l^2+k^2r^2}(l\mathbf{e_z}-kr\mathbf{e_\phi}), \tag{51}$$

where $\Psi_1^2<1$ and $\overline{\mathbf{B}}$ is given by Eq. (49),

$$\mathbf{v}=\sqrt{1-\Psi_1^2}\left(\frac{\mathbf{B}}{\rho}\frac{dF}{d\Psi}-\frac{d\Phi}{d\Psi}\frac{l\mathbf{e_z}-kr\mathbf{e_\phi}}{(l^2+k^2r^2)^2}\right), \tag{52}$$

$$\mathbf{E}=-\frac{d\Phi}{d\Psi}[(\frac{1}{4}(4b_1k^2r+2k^2r(l\Phi+kz)(2a_1-c_1(l\Phi+kz))+\frac{1}{2}c_1k^2r(l\Phi+kz)^2+\frac{2l^2}{r}(2b_1+2a_1(l\Phi+kz)))\mathbf{e_r}+(a_2l+\frac{1}{2}a_1k^2lr^2+4a_1l^3\ln(r))\mathbf{e_\phi}+(a_2k+\frac{1}{2}a_1k^3r^2+4a_1kl^2\ln(r))\mathbf{e_z}], \tag{53}$$

$$\mathbf{j} = \left[\left(\frac{\pounds\Psi_1}{\sqrt{1-\Psi_1^2}} + \frac{\Psi_1}{(1-\Psi_1^2)^{3/2}}|\nabla\Psi_1|^2\right) - 2klh^2 I\right]\frac{l\mathbf{e}_z - kr\mathbf{e}_\phi}{l^2+k^2r^2} - \frac{dI}{d\Psi}\overline{\mathbf{B}}, \tag{54}$$

$$P = p_s(\Psi_1) - \sqrt{1-\Psi_1^2}\left[\frac{\sqrt{1-\Psi_1^2}}{2}\left(\frac{B^2}{\rho}(\frac{dF}{d\Psi})^2 + \rho(l^2+k^2r^2)(\frac{d\Phi}{d\Psi})^2\right) + \theta\frac{d\Phi}{d\Psi}\right], \tag{55}$$

with

$$B^2 = \frac{1}{(l^2+k^2r^2)(1-\Psi^2)}[(\frac{a_2(l^2+k^2r)}{l^2+k^2r^2} + \frac{1}{2}a_1k^2r^2(l^3+k^2r) + 4a_1l^2(l^2+k^2r))^2 + (\frac{1}{4}(4b_1k^2r + 2k^2r(l\Phi+kz)(2a_1 - c_1(l\Phi+kz)) + \frac{1}{2}c_1k^2r(l\Phi+kz)^2 + \frac{2l^2}{r}(2b_1 + 2a_1(l\Phi+kz)))^2] + \frac{I^2}{l^2+k^2r^2}, \tag{56}$$

$$\pounds\Psi_1 = \frac{-6k^2l^2(b_1+a_1u)}{l^2+k^2r^2}, \tag{57}$$

and

$$|\nabla\Psi_1|^2 = \frac{1}{r^2}(4l^2+k^2r^2)^2(b_1+ka_1z+a_1l\phi)^2 + (a_2k + \frac{1}{2}k^3r^2 + 4a_1kl^2\ln(r))^2 + (a_2l + \frac{1}{2}k^2a_1lr^2 + 4a_1l^3\ln(r))^2. \tag{58}$$

Case 2. Put

$$-2lk\frac{X}{\sqrt{1-M^2}} = q_0\Psi, \quad \frac{1}{2}\frac{d}{d\Psi}\left(\frac{X^2}{1-M^2}\right) = q_1\Psi, \quad \frac{d}{d\Psi}\left(P_s - X\frac{dF}{d\Psi}\frac{d\Phi}{d\Psi}\right) = q_2\Psi, \quad \frac{d\rho}{d\Psi} = q_3\Psi, \ \frac{1}{2}\frac{d}{d\Psi}\left(\rho\left(\frac{d\Phi}{d\Psi}\right)^2\right) = q_4\Psi \tag{59}$$

in Eq. (26) we get the PDE:

$$\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r}\left(\frac{l^2-k^2r^2}{l^2+k^2r^2}\right)\frac{\partial\Psi}{\partial r} + \left(\frac{l^2+k^2r^2}{r^2}\right)\frac{\partial^2\Psi}{\partial u^2} + [\left(\frac{1}{l^2+k^2r^2}\right)q_0 + q_1(l^2+k^2r^2)(q_2 - q_3\Omega) + (l^2+k^2r^2)^2q_4]\Psi = 0, \tag{60}$$

where q_0, q_1, q_2, $q_3 \neq 0$ and q_4 are constants. We restrict q_3 of being not equal to zero, according (66) below.

To seek exact solutions to Eq. (60), we consider the following new variable [40]:

$$\Psi(r,u) = \alpha(r,u) + u\beta(r,u). \tag{61}$$

Substituting this form into Eq. (60) and equating the coefficients of like powers of u we get

$$\beta\left(\frac{q_0}{l^2+k^2r^2}+q_1+(l^2+k^2r^2)(q_2-q_3\Omega)+(l^2+k^2r^2)^2q_4\right)+\frac{(l^2+k^2r^2)}{r^2}\beta_{uu}+\frac{1}{r}\left(\frac{l^2-k^2r^2}{l^2+k^2r^2}\right)\beta_r+\beta_{rr}=0, \tag{62}$$

$$\alpha\left(\frac{q_0}{l^2+k^2r^2}+q_1+(l^2+k^2r^2)(q_2-q_3\Omega)+(l^2+k^2r^2)^2q_4\right)+\frac{2(l^2+k^2r^2)}{r^2}\beta_u+\frac{(l^2+k^2r^2)}{r^2}\alpha_{uu}+\frac{1}{r}\left(\frac{l^2-k^2r^2}{l^2+k^2r^2}\right)\alpha_r+\alpha_{rr}=0. \tag{63}$$

We consider the case $\alpha=\alpha(r), \beta=\beta(r)$ where $\alpha=\beta\neq 0$. For this case, we have

$$\frac{q_0}{l^2+k^2r^2}+q_1+(l^2+k^2r^2)(q_2-q_3\Omega)+(l^2+k^2r^2)^2q_4=\frac{1}{r\beta(r)}\left(\frac{l^2-k^2r^2}{l^2+k^2r^2}\right). \tag{64}$$

Substituting this form into Eq. (62) and (63) then solve them, we get

$$\beta(r)=\alpha(r)=\kappa_2-r+\frac{1}{2}\kappa_1k^2r^2+\kappa_1l^2\ln(r). \tag{65}$$

From Eq. (64)-(65), the gravitational potential is obtained as:

$$\Omega=\frac{1}{q_3}\left[\frac{q_0}{(l^2+k^2r^2)^2}+\frac{q_1}{l^2+k^2r^2}+(l^2+k^2r^2)q_4-\frac{l^2-k^2r^2}{r(l^2+k^2r^2)^2(\kappa_2-r+\frac{1}{2}\kappa_1k^2r^2+\kappa_1l^2\ln(r))}\right], \tag{66}$$

where κ_1, κ_2, are integration constants provided that the denominator of the last term at the right part of Eq. (66) is not equal to zero.

Thus, we get a solution for Eq. (60) as:

$$\Psi(r,u)=(1+u)\left(\kappa_2-r+\frac{1}{2}\kappa_1k^2r^2+\kappa_1l^2\ln(r)\right). \tag{67}$$

Using Eq. (7), solution (67) takes the form:

$$\Psi(r,\phi,z)=(1+l\phi+kz)\left(\kappa_2-r+\frac{1}{2}\kappa_1k^2r^2+\kappa_1l^2\ln(r)\right). \tag{68}$$

Using solution fo Eq. (68) to obtain the quantity $\overline{\mathbf{B}}$ as

$$\overline{\mathbf{B}}=-\frac{1}{r}(\kappa_2-r+\frac{1}{2}\kappa_1k^2r^2+\kappa_1l^2\ln(r))\mathbf{e_r}+(1+kz+l\phi)(-1+\frac{\kappa_1l^2}{r^2}+\kappa_1k^2r)\frac{l\mathbf{e_\phi}+kr\mathbf{e_z}}{l^2+k^2r^2}. \tag{69}$$

From Table 1 if we choose $M=\sin(\psi)$, then $\psi_2=\arcsin(\Psi_2)$. Hence we obtain an exact solution to Eq. (21) as follows:

$$\psi_2=\arcsin\left[(1+l\phi+kz)^2\left(c_2+c_1\left(\frac{k^2r^2}{2}+l^2\ln(r)\right)\right)\right]. \tag{70}$$

Using Eq. (31)-(35), the magnetic field, velocity field, electric field, vector of displacement current and gas pressure are obtained as:

$$\mathbf{B} = \frac{\bar{\mathbf{B}}}{\sqrt{1-\Psi_2^2}} + \frac{I}{l^2+k^2r^2}(l\mathbf{e_z} - kr\mathbf{e_\phi}), \tag{71}$$

where $\Psi_2^2 < 1$ and $\bar{\mathbf{B}}$ is given by Eq. (69),

$$\mathbf{v} = \sqrt{1-\Psi_2^2}\left(\frac{\mathbf{B}}{\rho}\frac{dF}{d\Psi} - \frac{d\Phi}{d\Psi}\frac{l\mathbf{e_z}-kr\mathbf{e_\phi}}{(l^2+k^2r^2)^2}\right), \tag{72}$$

$$\mathbf{E} = -\frac{d\Phi}{d\Psi}[(1+kz+l\phi)(-1+\frac{\kappa_1 l^2}{r^2}+\kappa_1 k^2 r)e_r + (\kappa_2 - r + \frac{1}{2}\kappa_1 k^2 r^2 + \kappa_1 l^2 \ln(r))(\frac{l}{r}\mathbf{e_\phi} + k\mathbf{e_z})], \tag{73}$$

$$\mathbf{j} = \left[\left(\frac{£\Psi_2}{\sqrt{1-\Psi_2^2}} + \frac{\Psi_2}{(1-\Psi_2^2)^{3/2}}|\nabla\Psi_2|^2\right) - 2klh^2 I\right]\frac{l\mathbf{e_z}-kr\mathbf{e_\phi}}{l^2+k^2r^2} - \frac{dI}{d\Psi}\bar{\mathbf{B}}, \tag{74}$$

$$P = p_s(\Psi_2) - \sqrt{1-\Psi_2^2}\left[\frac{\sqrt{1-\Psi_2^2}}{2}\left(\frac{B^2}{\rho}(\frac{dF}{d\Psi})^2 + \rho(l^2+k^2r^2)(\frac{d\Phi}{d\Psi})^2\right) + \theta\frac{d\Phi}{d\Psi}\right], \tag{75}$$

with

$$B^2 = \frac{1}{1-\Psi_2^2}[\frac{1}{r^2}(\kappa_2 - r + \frac{1}{2}\kappa_1 k^2 r^2 + \kappa_1 l^2 \ln(r))^2 + \frac{1}{l^2+k^2r^2}(1+kz+l\phi)^2(-1+\frac{\kappa_1 l^2}{r^2}+\kappa_1 k^2 r)^2] + \frac{I^2}{l^2+k^2r^2}, \tag{76}$$

$$£\Psi_2 = \frac{(1+kz+l\phi)(k^2r(-2\kappa_1 l^2 + r) + l^2(-1+2\kappa_1 k^2 r))}{r(l^2+k^2r^2)}, \tag{77}$$

and

$$|\nabla\Psi_2|^2 = (1+kz+l\phi)^2(-1+\frac{\kappa_1 l^2}{r^2}+\kappa_1 k^2 r)^2 + (\kappa_2 - r + \frac{1}{2}\kappa_1 k^2 r^2 + \kappa_1 l^2 \ln(r))^2(\frac{l^2}{r^2}k^2). \tag{78}$$

Case 3: Put

$$-2lk\frac{X}{\sqrt{1-M^2}} = d_1, \qquad \frac{d}{d\Psi}\left(P_s - X\frac{dF}{d\Psi}\frac{d\Phi}{d\Psi}\right) = d_2, \qquad \frac{d\rho}{d\Psi} = d_3\Psi^2, \qquad \frac{1}{2}\frac{d}{d\Psi}\left(\rho\left(\frac{d\Phi}{d\Psi}\right)^2\right) = d_4\Psi^2 \tag{79}$$

In Eq. (26) we get the PDE:

$$\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r}\left(\frac{l^2-k^2r^2}{l^2+k^2r^2}\right)\frac{\partial\Psi}{\partial r} + \left(\frac{l^2+k^2r^2}{r^2}\right)\frac{\partial^2\Psi}{\partial u^2} + \left(\frac{1}{l^2+k^2r^2}\right)d_1 + (l^2+k^2r^2)d_2 + (-d_3(l^2+k^2r^2)\Omega + (l^2+k^2r^2)^2 d_4)\Psi^2 = 0, \tag{80}$$

where d_1, d_2, $d_3 \neq 0$ and d_4 are constants. We restrict d_3 of being not equal to zero, according Eq. (84) below.

Using Eq. (61) into Eq. (80) and equating the coefficients of like powers of u, we get

$$u^2: \beta^2(-(l^2+k^2r^2)d_3\Omega + (l^2+k^2r^2)^2 d_4) = 0, \tag{81}$$

$$u^1: 2\beta\alpha(-(l^2+k^2r^2)d_3\Omega + (l^2+k^2r^2)^2 d_4) + \frac{(l^2+k^2r^2)}{r^2}\beta_{uu} + \frac{1}{r}\left(\frac{l^2-k^2r^2}{l^2+k^2r^2}\right)\beta_r + \beta_{rr} = 0, \tag{82}$$

$$u^0: \left[\frac{d_1}{l^2+k^2r^2} + (l^2+k^2r^2)d_2\right] + \alpha^2(-(l^2+k^2r^2)d_3\Omega + (l^2+k^2r^2)^2 d_4) + \frac{2(l^2+k^2r^2)}{r^2}\beta_u + \frac{(l^2+k^2r^2)}{r^2}\alpha_{uu} + \frac{1}{r}\left(\frac{l^2-k^2r^2}{l^2+k^2r^2}\right)\alpha_r + \alpha_{rr} = 0. \tag{83}$$

Eq. (81) gives

$$\Omega = \frac{d_4}{d_3}(l^2+k^2r^2). \tag{84}$$

We put

$$\beta_{uu} = 0, \tag{85}$$

and solve Eq. (82) and (85) to get:

$$\beta(r,u) = (1+u)\left(\chi_2 + \chi_1\left[\frac{k^2r^2}{2} + l^2\ln(r)\right]\right). \tag{86}$$

Also, put

$$\alpha_{uu} = -2\beta_u, \tag{87}$$

and solve Eq. (83) with using of Eq. (81), (84) and (86) in Eq. (87), we get

$$\alpha(r,u) = (u-u^2)\left(\chi_2 + \chi_1\left[\frac{k^2r^2}{2} + l^2\ln(r)\right]\right) + m_2 - \frac{1}{8}d_2k^2r^4 + m_0r^2 + m_1\ln(r). \tag{88}$$

We use $m_0 = 1/2(-l^2d_2 + k^2\chi_1)$ and $m_1 = \frac{1}{2k^2}(d_1 - l^4d_2 + 2k^2l^2\chi_1)$ where χ_1, χ_2 and m_2 are integration constants.

Therefore, using Eq. (88) we get a solution for Eq. (80) and (26) as:

$$\Psi(r,u) = 2u\left(\chi_2 + \chi_1\left[\frac{k^2r^2}{2} + l^2\ln(r)\right]\right) + m_2 - \frac{1}{8}d_2k^2r^4 + m_0r^2 + m_1\ln(r). \tag{89}$$

Using Eq. (7) solution (89) takes the form:

$$\Psi(r,\phi,z) = 2(l\phi + kz)\left(\chi_2 + \chi_1\left[\frac{k^2r^2}{2} + l^2\ln(r)\right]\right) + m_2 - \frac{1}{8}d_2k^2r^4 + m_0r^2 + m_1\ln(r). \tag{90}$$

Using the solution in Eq. (90) to obtain the quantity $\overline{\mathbf{B}}$ as:

$$\bar{\mathbf{B}} = -\frac{2}{r}(\chi_2 + \chi_1(\frac{k^2r^2}{2} + l^2\ln(r)))\mathbf{e_r} + [\frac{m_1}{r} + 2m_0r - \frac{1}{2}d_2k^2r^3 + \frac{\chi_1}{r}(l^2k^2r^2)(kz + l\phi)](\frac{l\mathbf{e_\phi} + kr\mathbf{e_z}}{l^2+k^2r^2}). \quad (91)$$

For $M = \sin(\psi)$ as shown in Table 1, $\psi_3 = \arcsin(\Psi_3)$. Hence we obtain an exact solution to Eq. (26) as follows:

$$\psi_3 = \arcsin\left[2(\phi + kz)\left(\chi_2 + \chi_1\left[\frac{k^2r^2}{2} + l^2\ln(r)\right]\right) + m_2 - \frac{1}{8}d_2k^2r^4 + m_0r^2 + m_1\ln(r)\right]. \quad (92)$$

The other physical quantities are obtained using Eq. (31)-(35) as:

$$\mathbf{B} = \frac{\bar{\mathbf{B}}}{\sqrt{1-\Psi_3^2}} + \frac{I}{l^2+k^2r^2}(l\mathbf{e_z} - kr\mathbf{e_\phi}), \quad (93)$$

where $\Psi_3^2 < 1$ and $\bar{\mathbf{B}}$ is given by Eq. (91),

$$\mathbf{v} = \sqrt{1-\Psi_3^2}\left(\frac{\mathbf{B}}{\rho}\frac{dF}{d\Psi} - \frac{d\Phi}{d\Psi}\frac{l\mathbf{e_z} - kr\mathbf{e_\phi}}{(l^2+k^2r^2)^2}\right), \quad (94)$$

$$\mathbf{E} = -\frac{d\Phi}{d\Psi}\{\left[\frac{m_1}{r} + 2m_0r - \frac{1}{2}d_2k^2r^3 + \frac{\chi_1}{r}(l^2 + k^2r^2)(kz + l\phi)\right]\mathbf{e_r} + \left[\chi_2 + \chi_1\left(\frac{k^2r^2}{2} + l^2\ln(r)\right)\right]\left(\frac{2l}{r}\mathbf{e_\Phi} + 2k\mathbf{e_z}\right)\}, \quad (95)$$

$$\mathbf{j} = \left[\left(\frac{£\Psi_3}{\sqrt{1-\Psi_3^2}} + \frac{\Psi_3}{(1-\Psi_3^2)^{3/2}}|\nabla\Psi_3|^2\right) - 2klh^2I\right]\frac{l\mathbf{e_z} - kr\mathbf{e_\phi}}{l^2+k^2r^2} - \frac{dI}{d\Psi}\bar{\mathbf{B}} \quad (96)$$

$$P = p_s(\Psi_3) - \sqrt{1-\Psi_3^2}\left\{\frac{\sqrt{1-\Psi_3^2}}{2}\left[\frac{B^2}{\rho}\left(\frac{dF}{d\Psi}\right)^2 + \rho(l^2 + k^2r^2)\left(\frac{d\Phi}{d\Psi}\right)^2\right] + \theta\frac{d\Phi}{d\Psi}\right\}, \quad (97)$$

with

$$B^2 = \frac{1}{1-\Psi^2}[\frac{4}{r^2}[\chi_2 + \chi_1(\frac{k^2r^2}{2} + l^2\ln(r))]^2 + \frac{1}{l^2+k^2r^2}[\frac{m_1}{r} + 2m_0r - \frac{1}{2}d_2k^2r^3 + \frac{\chi_1}{r}(l^2 + k^2r^2)(kz + l\phi)]^2 + \frac{I^2}{l^2+k^2r^2}], \quad (98)$$

$$£\Psi_3 = \frac{4l^2m_0 - 2k^2m_1 - 2d_2k^2l^2r^2 - d_2k^4r^4}{l^2+k^2r^2}, \quad (99)$$

and

$$|\nabla\Psi_3|^2 = \left[\frac{m_1}{r} + 2m_0r - \frac{1}{2}d_2k^2r^3 + \frac{\chi_1}{r}(l^2 + k^2r^2)(kz + l\phi)\right]^2 + \left[\chi_2 + \chi_1\left(\frac{k^2r^2}{2} + l^2\ln(r)\right)\right]^2\left(\frac{4l^2}{r^2} + 4k^2\right). \quad (100)$$

Case 4: Put

$$-2lk\frac{X}{\sqrt{1-M^2}}=A_0,\qquad \frac{d}{d\Psi}\left(P_s-X\frac{dF}{d\Psi}\frac{d\Phi}{d\Psi}\right)=A_1,\qquad \frac{d\rho}{d\Psi}=A_2,$$
$$\frac{1}{2}\frac{d}{d\Psi}\left(\rho\left(\frac{d\Phi}{d\Psi}\right)^2\right)=\frac{1}{2}A_2^3 \tag{101}$$

in Eq. (26) we get the PDE:

$$\frac{\partial^2\Psi}{\partial r^2}+\frac{1}{r}\left(\frac{l^2-k^2r^2}{l^2+k^2r^2}\right)\frac{\partial\Psi}{\partial r}+\left(\frac{l^2+k^2r^2}{r^2}\right)\frac{\partial^2\Psi}{\partial u^2}+\left(\frac{1}{l^2+k^2r^2}\right)A_0+$$
$$(l^2+k^2r^2)(A_1-A_2\Omega)+\frac{1}{2}(l^2+k^2r^2)^2A_2^3=0, \tag{102}$$

where A_0 A_1, and $A_2\neq 0$ are constants. We restrict A_2 of being not equal to zero, according (108) below.

To seek exact solutions to Eq. (102), we consider the following new variable

$$\Psi(r,u)=u^2\beta(r,u). \tag{103}$$

Substituting this form into Eq. (102) and equating the coefficients of like powers of u we get

$$u^2:(l^2+k^2r^2)^2\beta_{uu}+r(l^2-k^2r^2)\beta_r+r^2(l^2+k^2r^2)\beta_{rr}=0, \tag{104}$$

$$u^1:4(l^2+k^2r^2)^2\beta_u=0, \tag{105}$$

$$u^0:2(l^2+k^2r^2)^2\beta+r^2A_0+\frac{r^2}{2}(l^2+k^2r^2)^3A_2^3+r^2(l^2+k^2r^2)^2(A_1-A_2\Omega)=0. \tag{106}$$

Substitution of Eq. (105) into Eq. (104) yields

$$\beta(r)=c_2+c_1\left(\frac{k^2r^2}{2}+l^2\ln(r)\right). \tag{107}$$

Substitution of Eq. (107) into Eq. (106) gives

$$\Omega=\frac{2(l^2+k^2r^2)^2\left[c_2+c_1\left(\frac{k^2r^2}{2}+l^2\ln(r)\right)\right]+r^2A_0+\frac{r^2}{2}(l^2+k^2r^2)^3A_2^3+r^2(l^2+k^2r^2)^2A_1}{r^2(l^2+k^2r^2)^2A_2}, \tag{108}$$

where c_1 and c_2, a_1 are integral constants.

Thus, we get a solution for Eq. (103) as:

$$\Psi(r,u)=\left[c_2+c_1\left(\frac{k^2r^2}{2}+l^2\ln(r)\right)\right]u^2. \tag{109}$$

Using Eq. (7) the solution of Eq. (26) takes the form:

$$\Psi(r,\phi,z)=(l\phi+kz)^2\left[c_2+c_1\left(\frac{k^2r^2}{2}+l^2\ln(r)\right)\right]. \tag{110}$$

Using the solution of Eq. (110) to obtain the quantity $\overline{\mathbf{B}}$ as:

$$\overline{\mathbf{B}} = \frac{-2}{r}(l\phi + kz)\left[c_2 + c_1\left(\frac{k^2r^2}{2} + l^2\ln(r)\right)\right]\mathbf{e}_r + c_1(l\phi + kz)^2(lr\mathbf{e}_\phi + kr^2\mathbf{e}_z). \tag{111}$$

From Table 1 if we choose $M = \sin(\psi)$, then $\psi = \arcsin(\Psi)$. Hence we obtain an exact solution to Eq. (21) as:

$$\psi_4 = \arcsin\left[(l\phi + kz)^2\left(c_2 + c_1\left(\frac{k^2r^2}{2} + l^2\ln(r)\right)\right)\right]. \tag{112}$$

The other physical variables are obtained using Eq. (31)-(35) as follows:

$$\mathbf{B} = \frac{\overline{\mathbf{B}}}{\sqrt{1-\Psi_4^2}} + \frac{I}{l^2+k^2r^2}(l\mathbf{e_z} - kr\mathbf{e_\phi}), \tag{113}$$

where $\Psi_4^2 < 1$, and $\overline{\mathbf{B}}$ is given by Eq. (111),

$$\mathbf{v} = \sqrt{1-\Psi_4^2}\left(\frac{\mathbf{B}}{\rho}\frac{dF}{d\Psi} - \frac{d\Phi}{d\Psi}\frac{l\mathbf{e_z} - kr\mathbf{e_\phi}}{(l^2+k^2r^2)^2}\right), \tag{114}$$

$$\mathbf{E} = -\frac{d\Phi}{d\Psi}\{c_1r(l^2 + k^2r^2)(l\phi + kz)^2\mathbf{e_r} + (l\phi + kz)\left[c_2 + c_1\left(\frac{k^2r^2}{2} + l^2\ln(r)\right)\right] \times \left(\frac{2l}{r}\mathbf{e_\phi} + 2k\mathbf{e_z}\right)\}, \tag{115}$$

$$\mathbf{j} = \left[\left(\frac{\pounds\Psi_4}{\sqrt{1-\Psi_4^2}} + \frac{\Psi_4}{(1-\Psi_4^2)^{3/2}}|\nabla\Psi_4|^2\right) - 2klh^2I\right]\frac{l\mathbf{e_z} - kr\mathbf{e_\phi}}{l^2+k^2r^2} - \frac{dI}{d\Psi}\overline{\mathbf{B}}, \tag{116}$$

$$P = p_s(\Psi) - \sqrt{1-\Psi_4^2}\left[\frac{\sqrt{1-\Psi_4^2}}{2}\left(\frac{B^2}{\rho}\left(\frac{dF}{d\Psi}\right)^2 + \rho(l^2 + k^2r^2)\left(\frac{d\Phi}{d\Psi}\right)^2\right) + \theta\frac{d\Phi}{d\Psi}\right], \tag{117}$$

with

$$B^2 = \frac{4(l\phi+kz)^2}{r^2(1-\Psi_4^2)}\left[c_2 + c_1\left(\frac{k^2r^2}{2} + l^2\ln(r)\right)\right]^2 + \frac{c_1^2l^2r^2 + c_1^2k^2r^4}{1-\Psi_4^2}(l\phi + kz)^4 + \frac{I^2}{l^2+k^2r^2}, \tag{118}$$

$$\pounds\Psi_4 = \frac{2(l^2+k^2r^2)}{r^2}\left[c_2 + c_1\left(\frac{k^2r^2}{2} + l^2\ln(r)\right)\right], \tag{119}$$

and

$$|\nabla\Psi_4|^2 = c_1^2(l\phi + kz)^4\left(k^2r + \frac{l^2}{r}\right)^2 + (l\phi + kz)^2\left(4k^2 + \frac{4l^2}{r^2}\right)[c_2 + c_1(\frac{k^2r^2}{2} + l^2\ln(r))]^2. \tag{120}$$

Case 5: Put

$$-2lk\frac{X}{\sqrt{1-M^2}} = d_1,\ \frac{d}{d\Psi}\left(P_s - X\frac{dF}{d\Psi}\frac{d\Phi}{d\Psi}\right) = d_2,\ \frac{d\rho}{d\Psi} = d_3\Psi,\ \frac{1}{2}\frac{d}{d\Psi}\left(\rho\left(\frac{d\Phi}{d\Psi}\right)^2\right) = \frac{1}{2}d_3^3\Psi, \tag{121}$$

in Eq. (26) we get the PDE:

$$\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r}\left(\frac{l^2 - k^2r^2}{l^2 + k^2r^2}\right)\frac{\partial\Psi}{\partial r} + \left(\frac{l^2 + k^2r^2}{r^2}\right)\frac{\partial^2\Psi}{\partial u^2} + \left(\frac{1}{l^2 + k^2r^2}\right)d_1 + (l^2 + k^2r^2)d_2 + \left(-(l^2 + k^2r^2)\Omega + \frac{1}{2}(l^2 + k^2r^2)^2\right)d_3\Psi = 0, \tag{122}$$

where d_1, $d_2 \neq 0$ and d_3 are constants. We restrict d_2 of being not equal to zero, according (127) below.

Substituting Eq. (103) into Eq. (122) and equating the coefficients of like powers of u, we get

$$u^2: r^2\beta\left(-(l^2 + k^2r^2)^2 d_3\Omega + \frac{1}{2}(l^2 + k^2r^2)^3 d_3^3\right) + (l^2 + k^2r^2)^2\beta_{uu} r\left(\frac{l^2 - k^2r^2}{l^2 + k^2r^2}\right)\beta_r + r^2(l^2 + k^2r^2)\beta_{rr} = 0, \tag{123}$$

$$u^1: \beta_u = 0, \tag{124}$$

$$u^0: 2(l^2 + k^2r^2)^2\beta + r^2 d_1 + r^2(l^2 + k^2r^2)^2 d_2 = 0. \tag{125}$$

Eq. (125) gives

$$\beta = \frac{-r^2d_1 - r^2(l^2+k^2r^2)^2 d_2}{2(l^2+k^2r^2)^2}. \tag{126}$$

Substitution of Eq. (126) and (124) into Eq. (123) yields

$$\Omega = \frac{\frac{4r^2(l^4 - 5l^2k^2r^2 + 2k^4r^4)d_1}{(l^2+k^2r^2)^3} + 4l^2r^2d_2 + \frac{1}{2}d_3^3 r^4(l^2+k^2r^2)(d_1 + (l^2+k^2r^2)^2 d_2)}{r^4(d_1 + (l^2+k^2r^2)^2)d_2}. \tag{127}$$

Therefore, we get a solution for Eq. (122) and (26) as:

$$\Psi(r,u) = u^2\left(\frac{-r^2d_1 - r^2(l^2+k^2r^2)^2 d_2}{2(l^2+k^2r^2)^2}\right), \tag{128}$$

$$\Psi(r,\phi,z) = (l\phi + kz)^2\left(\frac{-r^2d_1 - r^2(l^2+k^2r^2)^2 d_2}{2(l^2+k^2r^2)^2}\right). \tag{129}$$

Using solution in Eq. (129) to obtain the quantity $\overline{\mathbf{B}}$ as:

$$\overline{\mathbf{B}} = \frac{r(d_1 + d_2(l^2+k^2r^2))(l\phi+kz)}{(l^2+k^2r^2)^3}\left[(l^2 + k^2r^2)\mathbf{e}_r - l(kz + l\phi)\mathbf{e}_\phi - kr(kz + l\phi)\mathbf{e}_z\right]. \tag{130}$$

For $M = \sin(\psi)$ as shown in Table 1, $\psi = \arcsin(\Psi_5)$. Hence we obtain an exact solution to Eq. (21) as follows:

$$\psi_5 = \arcsin\left[(l\phi + kz)^2\left(\frac{-r^2 d_1 - r^2(l^2+k^2r^2)^2 d_2}{2(l^2+k^2r^2)^2}\right)\right]. \tag{131}$$

The other physical variables are obtained as follows:

$$\mathbf{B} = \frac{\overline{\mathbf{B}}}{\sqrt{1-\Psi_5^2}} + \frac{I}{l^2+k^2r^2}(l\mathbf{e_z} - kr\mathbf{e_\phi}), \tag{132}$$

where $\Psi_5^2 < 1$ and $\overline{\mathbf{B}}$ is given by Eq. (130),

$$\mathbf{v} = \sqrt{1-\Psi_5^2}\left(\frac{\mathbf{B}}{\rho}\frac{dF}{d\Psi} - \frac{d\Phi}{d\Psi}\frac{l\mathbf{e_z} - kr\mathbf{e_\phi}}{(l^2+k^2r^2)^2}\right), \tag{133}$$

$$\mathbf{E} = -\frac{d\Phi}{d\Psi}\left[\frac{(d_1 + d_2(l^2+k^2r^2))(l\phi+kz)}{(l^2+k^2r^2)^2}(-r(l\phi+kz)\mathbf{e_r} - lr\mathbf{e_\phi} - kr^2\mathbf{e_z})\right], \tag{134}$$

$$\mathbf{j} = \left[\left(\frac{\pounds\Psi_5}{\sqrt{1-\Psi_5^2}} + \frac{\Psi_5}{(1-\Psi_5^2)^{3/2}}|\nabla\Psi_5|^2\right) - 2klh^2 I\right]\frac{l\mathbf{e_z} - kr\mathbf{e_\phi}}{l^2+k^2r^2} - \frac{dI}{d\Psi}\overline{\mathbf{B}} \tag{135}$$

$$P = p_s(\Psi_5) - \sqrt{1-\Psi_5^2}\left[\frac{\sqrt{1-\Psi_5^2}}{2}\left(\frac{B^2}{\rho}\left(\frac{dF}{d\Psi}\right)^2 + \rho(l^2+k^2r^2)\left(\frac{d\Phi}{d\Psi}\right)^2\right) + \theta\frac{d\Phi}{d\Psi}\right], \tag{136}$$

with

$$B^2 = \frac{r^2(d_1 + d_2(l^2+k^2r^2))^2(l\phi+kz)^2}{(l^2+k^2r^2)^6(1-\Psi_5^2)}[(l^2+k^2r^2)^2 + l^2(kz+l\phi)^2 + k^2r^2(kz+l\phi)^2] + \frac{I^2}{l^2+k^2r^2}, \tag{137}$$

$$\pounds\Psi_5 = \frac{2d_2 l^2(l^2+k^2r^2)^3 + 2d_1(l^4 - 5k^2l^2r^2 + 2k^4r^4)}{(l^2+k^2r^2)^4}, \tag{138}$$

and

$$|\nabla\Psi_5|^2 = \frac{(d_1 + d_2(l^2+k^2r^2))^2(l\phi+kz)^2}{(l^2+k^2r^2)^4}(r^2(l\phi+kz)^2 + l^2r^2 + k^2r^4). \tag{139}$$

Thus, we have obtain exact solution classes to the whole ideal MHD system.

6. Conclusions

In this paper, the equilibrium equations of helically symmetric ideal MHD plasmas with incompressible flows in the presence of a variable gravitational field are derived. Self-similar transformations are proposed to deal with many nonlinear cases of the equilibrium equations. Several exact solutions to the whole ideal MHD system are obtained.

Previously, the equilibrium equations with incompressible flows are considered for an axisymmetric gravitating magnetically confined plasma [41-43]. Examples of MHD equilibria with constant Mach number were constructed [43], and analytical solutions were obtained [44] for a plasma confined to a dipolar magnetic field subject to massive body gravitational forces. In [43-44] it was shown that for the poloidal magnetic flux function, the MHD equilibrium states of an axisymmetric toroidal plasma with finite resistivity and flows parallel to the magnetic field are controlled by a second order nonlinear PDE [43] coupled with a Poisson's equation for the gravitational potential [44].

Here, we considered the problem in the case of helically symmetric domain that is more general than axisymmetric one. Moreover, we considered gravitating plasmas with variable Mach number and variable gravitational field. We have dealt with several nonlinear cases of the equations of motion for these plasmas and obtained exact solutions for them.

References

[1] Throumoulopoulos, G.N. and Tasso, H. "Magnetohydrodynamic equilibria with helical symmetry and incompressible flows." *J. Plasma Phys.* 62, no 4 (1999): 449-459. https://doi.org/10.1017/S0022377899008041
[2] Sarasua, L. G., AC Sicardi Schifino, and R. Gonzalez. "The stability of steady, helical vortex filaments in a tube." *Physics of Fluids* 11, no. 5 (1999): 1096-1103. https://doi.org/10.1063/1.869980
[3] Zabielski, L., and A. J. Mestel. "Steady flow in a helically symmetric pipe." *Journal of Fluid Mechanics* 370 (1998): 297-320. https://doi.org/10.1017/S0022112098002006
[4] Fukumoto, Yasuhide, and V. L. Okulov. "The velocity field induced by a helical vortex tube." *Physics of Fluids* 17, no. 10 (2005): 107101. https://doi.org/10.1063/1.2061427
[5] IJzermans, Rutger HA, Rob Hagmeijer, and Pieter J. van Langen. "Accumulation of heavy particles around a helical vortex filament." *Physics of fluids* 19, no. 10 (2007): 107102. https://doi.org/10.1063/1.2771658
[6] Nakamura, M., Y. Uchida, and S. Hirose. "Production of wiggled structure of AGN radio jets in the sweeping magnetic twist mechanism." *New Astronomy* 6, no. 2 (2001): 61-78. https://doi.org/10.1016/S1384-1076(01)00041-0
[7] Lovelace, R. V. E., H. Li, G. V. Ustyugova, M. M. Romanova, and S. A. Colgate. "Poynting jets from accretion disks." In *AIP Conference Proceedings*, vol. 586, no. 1, pp. 426-432. American Institute of Physics, 2001. https://doi.org/10.1063/1.1419587
[8] Hujeirat, A., M. Livio, M. Camenzind, and A. Burkert. "A model for the jet-disk connection in BH accreting systems." *Astronomy & Astrophysics* 408, no. 2 (2003): 415-430. https://doi.org/10.1051/0004-6361:20031040
[9] Tsinganos, K., and S. Bogovalov. "Magnetic collimation of relativistic outflows in jets with a high mass flux." *Monthly Notices of the Royal Astronomical Society* 337, no. 2 (2002): 553-558. https://doi.org/10.1046/j.1365-8711.2002.05933.x
[10] Ito, Hidesato. "Flow in curved pipes." *JSME international journal* 30, no. 262 (1987): 543-552. https://doi.org/10.1299/jsme1987.30.543
[11] Liu, Shijie, and Jacob H. Masliyah. "Axially invariant laminar flow in helical pipes with a finite pitch." *Journal of Fluid Mechanics* 251 (1993): 315-353. https://doi.org/10.1017/S002211209300343X
[12] Germano, M. "The Dean equations extended to a helical pipe flow." *Journal of Fluid Mechanics* 203 (1989): 289-305. https://doi.org/10.1017/S0022112089001473
[13] Tuttle, E. R. "Laminar flow in twisted pipes." *Journal of Fluid Mechanics* 219 (1990): 545-570. https://doi.org/10.1017/S002211209000307X
[14] Zabielski, L., and A. J. Mestel. "Unsteady blood flow in a helically symmetric pipe." *Journal of Fluid Mechanics* 370 (1998): 321-345. https://doi.org/10.1017/S0022112098001992
[15] Kuiroukidis, Ap. "Stable non-separable tokamak equilibria with parallel flows." *Plasma Physics and Controlled Fusion* 52, no. 1 (2009): 015002. https://doi.org/10.1088/0741-3335/52/1/015002
[16] Hothi, N., and S. Bisht. "Mesonic and baryonic Regge trajectories with quantized masses." *Indian Journal of Physics* 85, no. 12 (2011): 1833-1842.

https://doi.org/10.1007/s12648-011-0189-1
[17] Throumoulopoulos, G. N., and G. Pantis. "Magnetohydrodynamic equilibria of a cylindrical plasma with poloidal mass flow and arbitrary cross sectional shape." *Plasma physics and controlled fusion* 38, no. 10 (1996): 1817. https://doi.org/10.1088/0741-3335/38/10/009
[18] Simintzis, Ch, G. N. Throumoulopoulos, G. Pantis, and H. Tasso. "Analytic magnetohydrodynamic equilibria of a magnetically confined plasma with sheared flows." *Physics of Plasmas* 8, no. 6 (2001): 2641-2648. https://doi.org/10.1063/1.1371768
[19] Throumoulopoulos, G. N., G. Poulipoulis, G. Pantis, and H. Tasso. "Exact magnetohydrodynamic equilibria with flow and effects on the Shafranov shift." *Physics Letters A* 317, no. 5-6 (2003): 463-469. https://doi.org/10.1016/j.physleta.2003.09.005
[20] Throumoulopoulos, G. N., H. Tasso, and G. Poulipoulis. "Side-conditioned axisymmetric equilibria with incompressible flows." *Journal of plasma physics* 74, no. 3 (2008): 327-344. https://doi.org/10.1017/S0022377807006769
[21] Khater, A. H., and S. M. Moawad. "Exact equilibria for nonlinear cylindrical ideal magnetohydrodynamic plasma with steady incompressible flow and arbitrary cross sectional shape." *Physics of Plasmas* 16, no. 12 (2009): 122506. https://doi.org/10.1063/1.3270522
[22] Throumoulopoulos, G. N., H. Tasso, and G. Poulipoulis. "Magnetohydrodynamic 'cat eyes' and stabilizing effects of plasma flow." *Journal of Physics A: Mathematical and Theoretical* 42, no. 33 (2009): 335501. https://doi.org/10.1088/1751-8113/42/33/335501
[23] Andreussi, T., P. J. Morrison, and F. R. A. N. C. E. S. C. O. Pegoraro. "MHD equilibrium variational principles with symmetry." *Plasma Physics and Controlled Fusion* 52, no. 5 (2010): 055001. https://doi.org/10.1088/0741-3335/52/5/055001
[24] Throumoulopoulos, G. N., and H. Tasso. "International thermonuclear experimental reactor-like extended Solovév equilibria with parallel flow." *Physics of Plasmas* 19, no. 1 (2012): 014504. https://doi.org/10.1063/1.3672509
[25] Moawad, S. M., and D. A. Ibrahim. "Three-dimensional nonlinear ideal MHD equilibria with field-aligned incompressible and compressible flows." *Physics of Plasmas* 23, no. 8 (2016): 082502. https://doi.org/10.1063/1.4960043
[26] Rizk, H. M. "Helical equilibrium with anisotropic pressure and mass flow." *Physica Scripta* 31, no. 6 (1985): 592. https://doi.org/10.1088/0031-8949/31/6/021
[27] Krasnopolsky, Ruben, Zhi-Yun Li, and Roger D. Blandford. "Magnetocentrifugal Launching of Jets from Accretion Disks. II. Inner Disk-driven Winds." *The Astrophysical Journal* 595, no. 2 (2003): 631. https://doi.org/10.1086/377494
[28] Ouyed, Rachid, David A. Clarke, and Ralph E. Pudritz. "Three-dimensional simulations of jets from Keplerian disks: Self-regulatory stability." *The Astrophysical Journal* 582, no. 1 (2003): 292. https://doi.org/10.1086/344507
[29] Casse, Fabien, and Rony Keppens. "Radiatively inefficient magnetohydrodynamic accretion-ejection structures." *The Astrophysical Journal* 601, no. 1 (2004): 90. https://doi.org/10.1086/380441
[30] Nakamura, Masanori, and David L. Meier. "Poynting Flux-dominated Jets in Decreasing-Density Atmospheres. I. The Nonrelativistic Current-driven Kink Instability and the Formation of "Wiggled" Structures." *The Astrophysical Journal* 617, no. 1 (2004): 123. https://doi.org/10.1086/425337
[31] Tsui, K. H., C. E. Navia, A. Serbeto, and H. Shigueoka. "Tokamak equilibria with non field-aligned axisymmetric divergence-free rotational flows." *Physics of Plasmas* 18, no. 7 (2011): 072502. https://doi.org/10.1063/1.3604946
[32] Hameiri, Eliezer. "The equilibrium and stability of rotating plasmas." *The Physics of Fluids* 26, no. 1 (1983): 230-237. https://doi.org/10.1063/1.864012
[33] Del Zanna, L., and C. Chiuderi. "Exact solutions for symmetric magnetohydrodynamic equilibria with mass flow." *Astronomy and Astrophysics* 310 (1996): 341-350.
[34] Kumar, N., P. Kumar, A. Kumar, and R. Chauhan. "Role of equilibrium plasma flow on damping of slow MHD waves." *Indian Journal of Physics* 85, no. 12 (2011): 1879-1886. https://doi.org/10.1007/s12648-011-0191-7
[35] Ershkov, Sergey V. "About existence of stationary points for the Arnold–Beltrami–Childress (ABC) flow." *Applied Mathematics and Computation* 276 (2016): 379-383.

https://doi.org/10.1016/j.amc.2015.12.038

[36] Ershkov, Sergey V., Roman V. Shamin, and Ayrat R. Giniyatullin. "On a new type of non-stationary helical flows for incompressible 3D Navier-Stokes equations." *Journal of King Saud University-Science* 32, no. 1 (2020): 459-467. https://doi.org/10.1016/j.jksus.2018.07.006

[37] Ershkov, Sergey V. "Non-stationary helical flows for incompressible 3D Navier–Stokes equations." *Applied Mathematics and Computation* 274 (2016): 611-614. https://doi.org/10.1016/j.amc.2015.11.039

[38] Moffatt, H. Keith. "Helicity and singular structures in fluid dynamics." *Proceedings of the National Academy of Sciences* 111, no. 10 (2014): 3663-3670. https://doi.org/10.1073/pnas.1400277111

[39] Wahid, Nur Syahirah, Mohd Ezad Hafidz Hafidzuddin, Norihan Md Arifin, Mustafa Turkyilmazoglu, and Nor Aliza Abd Rahmin. "Magnetohydrodynamic (MHD) Slip Darcy Flow of Viscoelastic Fluid Over A Stretching Sheet and Heat Transfer with Thermal Radiation and Viscous Dissipation." *CFD Letters* 12, no. 1 (2020): 1-12.

[40] Clarkson, Peter A., and Martin D. Kruskal. "New similarity reductions of the Boussinesq equation." *Journal of Mathematical Physics* 30, no. 10 (1989): 2201-2213. https://doi.org/10.1063/1.528613

[41] White, R. L., and R. D. Hazeltine. "Symmetry analysis of the Grad–Shafranov equation." *Physics of Plasmas* 16, no. 12 (2009): 123101. https://doi.org/10.1063/1.3267211

[42] Litvinenko, Yuri E. "A similarity reduction of the Grad–Shafranov equation." *Physics of Plasmas* 17, no. 7 (2010): 074502. https://doi.org/10.1063/1.3456519

[43] Throumoulopoulos, G.N. and Tasso, H. "On resistive magnetohydrodynamic equilibria of an axisymmetric troroidal plasma with flow." *J. Plasma Phys.* 64, no 5 (2000): 601-611. https://doi.org/10.1017/S0022377800008849

[44] Throumoulopoulos, G. N., and H. Tasso. "Axisymmetric equilibria of a gravitating plasma with incompressible flows." *Geophysical & Astrophysical Fluid Dynamics* 94, no. 3-4 (2001): 249-262. https://doi.org/10.1080/03091920108203409