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Relaxation Equations of Consolidating Cake Filtration

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ABSTRACT

In this paper, on the basis of conservation laws suspensions filtration equations with forming a relaxing cake are derived. The equations are numerically solved. To solve the equation for cake growth a Stefan problem is posed and solved with using the method of catching a moving front. On the basis of numerical results influence of relaxation phenomena on filtration characteristics is established. It is shown that the relaxation slows down the increasing of local filter cake porosity and decreasing of cake permeability, as well as increase fluid relative velocity through cake for given applied pressure. Increasing of relaxation time leads to a faster growth of the cake thickness when all other conditions are constants, thus relaxation effects cause more intensive transfer of particles from suspension to the cake. It, in turn, alters all other filtration characteristics, such as fluid pressure distribution, compressive pressure, porosity and consolidation of the cake, fluid flow rate through the cake and effective hydrodynamic resistance. With increasing of relaxation phenomena dynamics of current and total outlet filtrate flow rate becomes more intensive.

Keywords:

Cake filtration; filter; relaxation; relaxing cake; Stefan problem; suspension

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1. Introduction

Filtration refers to the processes of separation of heterogeneous systems using a porous medium (filter), which hold up one phase of these systems and let others pass through. These processes include separation of the suspension into pure liquid and solid particles. Processes of suspensions separation have wide applications in chemical, petrochemical, oil and gas, coal, food and some other industries. At filtration, solid particles contained in the suspension are retained on the surface of the filter and form a cake. This process is called as filtering with formation of a cake [1-7]. The cake formed during the filtration of different suspensions usually is a complex multi-component system consisting, in general, of liquid and solid particles of various shapes and sizes. Solid particles of the cake are under the influence of external and internal forces and interact with the surrounding medium and with each other. External fields influencing on cake particles include pressure drop and

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gravitational forces. Internal fields are excited by inter particle interaction forces. These include: the forces of chemical nature, molecular, ion-electrostatic, capillary and magnetic forces [8-10]. To study the influence of these factors is very important in the design and construction of different filtering elements, for instance, ultrafiltration membranes (UF). UF has been recognized as one of the advanced technologies and is widely implemented in various field of industry. The performance of sandwich membrane with different configurations is studied in [11]. Separation properties and performance in membrane reactor systems are studied in [12].

The filtration with cake formation is based on the fundamental equations of the mechanics of multi-component media, the theory of filtration consolidation [13-20] and takes into account the influence of hydrodynamic and rheological factors. Differential equations of filtering are obtained on the basis of the two-phase flows mechanics equations, taking into account the changes in the filter parameters along the thickness of the cake with the time [9,10,21,22]. Since the cake-suspension interface is mobile, the equation describing the compressibility of the cake leads to Stefan's problems [14,23,24].

When we consider non-Newtonian systems, such as, highly concentrated suspensions, as well as suspensions with high-molecular polymers, the classical Darcy's law is violated. In this case, the Darcy law is nonlinear or non-equilibrium. At filtration of dispersed systems in porous media due to deposition of some phase components in the pore space, the filtration flow can have relaxation behaviour. A number of works are devoted to this phenomenon [25-31]. In this case, we have a non-equilibrium relationship between the filtration velocity and the pressure gradient. Relaxation phenomena in the filtration of liquids are reflected in the equations used for mathematical modeling of the filtration process. Usually, in the description of relaxation filtration, various phenomenological models are used, taking into account the delay in the dependences between the filtration velocity and the pressure gradient. In some cases, it is necessary also to take into account the delay in the equations of state. In our case liquid phase of suspension, we treat as polymer solution that exhibits non-equilibrium rheological properties, as a consequence, it leads to a non-equilibrium filtration law. So, we use here relaxation type filtration law, where pressure gradient relaxes with respect the filtration velocity.

As one of the first in this direction, one can mention the work [32]. In this work on the basis of numerous experimental data, a relaxation changes in the velocity of filtration during the flow of a polymer solution through a porous medium at a constant pressure drop is shown. Khuzhayorov [31] research on some non-steady one-dimensional filtration problems under the elastic regime are considered with an assumption that the pressure gradient has relaxation behaviour with respect to the velocity of filtration. A systematic analysis of the relaxation theory of filtration is given in the research by Molokovich [34].

As far as we know, still the relaxation phenomena in the theory of cake filtration are not applied. However, it is important both in theoretical and practical point of view. The studies of relaxation phenomena allow us better understanding of suspensions separation processes when liquid phase of suspension has high viscosity and anomalous rheological properties.

In this paper, we consider only phenomenological models for the filtration of disperse systems. Macroscopic modeling has recently been very intensively developed, which yet is not practically used yet in filtration problems of disperse systems. Some macroscopic models of relaxation mass transport in a porous medium are derived and analyzed in the research by Khuzhayorov [35]. We first derive cake filtration equations and an equation for the cake growth. Then we perform numerical analyses of equations and present results. On the basis of the last we make some conclusions. Note, here we emphasize our attention only on the relaxation effects in the filtration law. In general, relaxation effects can have other nature, for instance cake properties can have non-equilibrium dependences.

2. Mathematical Model

2.1 Derivation of Cake Filtration Equations

We consider cake filtration of suspensions (Figure 1). At filtration of suspensions with forming a cake, the Darcy's law establishes an equilibrium relationship between the relative velocity q_{ls} and the pressure gradient $\partial p_l / \partial x$, which in one-dimensional scalar form can be written as [5]

$$\frac{q_{ls}}{\varepsilon} = \frac{q_l}{\varepsilon} - \frac{q_s}{\varepsilon} = -\frac{1}{\varepsilon} \frac{k}{\mu} \frac{\partial p_l}{\partial x},$$

where l is index corresponding to the liquid phase, s is index corresponding to solid particles, ε_i is porosity of the i -th phase (here and after we use ε instead of ε_l for simplicity), q_l is liquid phase velocity, q_s is solid phase velocity, p_l is pressure in the liquid phase, k is permeability coefficient, μ is viscosity.

Suppose that this relationship is of a non-equilibrium nature. Non-equilibrium nature can occur both in the filtration velocity of the liquid and solid phases relative to the pressure gradient, and vice versa, in the pressure gradient relative to the filtration velocities. The non-equilibrium relationship is assumed in the linear differential form. In addition, since the velocities of solid and liquid phases can have different scales of variation, the relaxation effects can also occur with different characteristic times.

Following to this idea the relaxation filtering law we take in the form

$$\left(1 + \lambda_{ql} \frac{\partial}{\partial t}\right) \frac{q_l}{\varepsilon} - \left(1 + \lambda_{qs} \frac{\partial}{\partial t}\right) \frac{q_s}{\varepsilon_s} = -\frac{1}{\varepsilon} \frac{k}{\mu} \left(1 + \lambda_{pl} \frac{\partial}{\partial t}\right) \frac{\partial p_l}{\partial x}, \quad (1)$$

where λ_{ql} is the relaxation time of filtration velocities, λ_{pl} is the relaxation time of the pressure gradient.

In the cake, the velocity of the particles is much lower than the velocity of the liquid phase. The velocity of particles at the boundary of the suspension and the cake can be assumed to be equal to the velocity of the suspension, and at the boundary of the cake and the filter surface, this velocity is zero. So, the filtration velocity of solid particles decreases along the cake from the velocity of the suspension to zero. The velocity of the liquid phase increases along the cake from the boundary with the suspension to the boundary of the filter, while the overall filtration velocity $q_l + q_s$ remains constant with respect to x . Because of this, the change rate in the velocity of filtration of the solid phase compared with the liquid phase can be considered to be slow. This circumstance makes it possible to neglect the relaxation effects in the filtration velocity of the solid phase in comparison with the liquid phase. Then Eq. (1) can be written in the form

$$\left(1 + \lambda_{ql} \frac{\partial}{\partial t}\right) \frac{q_l}{\varepsilon} - \frac{q_s}{\varepsilon_s} = -\frac{1}{\varepsilon} \frac{k}{\mu} \left(1 + \lambda_{pl} \frac{\partial}{\partial t}\right) \frac{\partial p_l}{\partial x}, \quad (2)$$

For the sake of simplicity, let us first consider only relaxation of the pressure gradient, i.e. we take $\lambda_{ql} = 0$. Then Eq. (1) takes the form

$$\frac{q_l}{\varepsilon} - \frac{q_s}{\varepsilon_s} = -\frac{1}{\varepsilon} \frac{k}{\mu} \left(1 + \lambda_{pl} \frac{\partial}{\partial t}\right) \frac{\partial p_l}{\partial x}. \quad (3)$$

At $x = 0$ by taking into account $q_s|_{x=0} = 0$ from Eq. (3) we have

$$q_l|_{x=0} = q_{lm} = -\frac{k}{\mu} \left(1 + \lambda_{pl} \frac{\partial}{\partial t} \right) \frac{\partial p_l}{\partial x} \Big|_{x=0}, \quad (4)$$

where $q_{lm} = 0$ is the instantaneous filtration velocity at $x = 0$. Because $q_s|_{x=0} = 0$, the term q_{lm} represents, in fact, filtrate velocity.

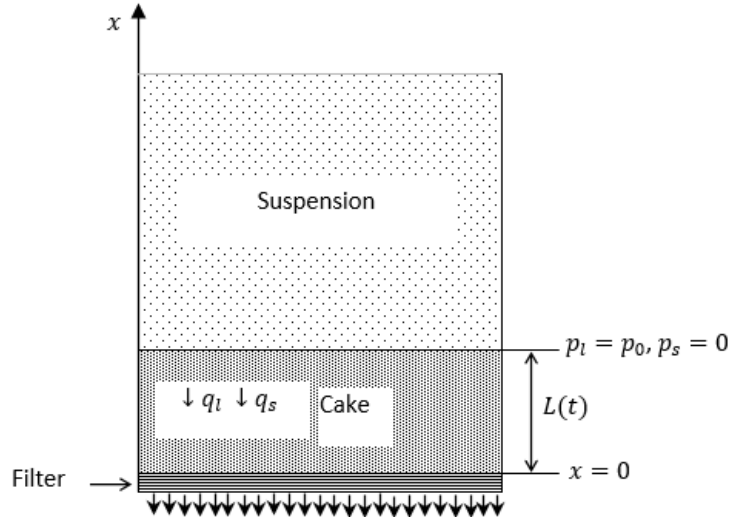


Fig. 1. A schematic of cake filtration

The flow q_{lm} is balanced by the flow through the filter, so in Eq. (4) we have

$$q_l|_{x=0} = -\frac{k}{\mu} \left(1 + \lambda_{pl} \frac{\partial}{\partial t} \right) \frac{\partial p_l}{\partial x} \Big|_{x=0} = -\frac{p_{lm}}{\mu R_m}, \quad (5)$$

where p_{lm} is the filtrate pressure at $x = 0$, R_m is relative resistance of the filtering element. We transform Eq. (3) into

$$\frac{q_l}{\varepsilon} - \frac{q_{lm} - q_l}{\varepsilon_s} = -\frac{1}{\varepsilon} \frac{k}{\mu} \left(1 + \lambda_{pl} \frac{\partial}{\partial t} \right) \frac{\partial p_l}{\partial x}$$

and from that we have

$$q_l = \varepsilon q_{lm} - \varepsilon_s \frac{k}{\mu} \left(1 + \lambda_{pl} \frac{\partial}{\partial t} \right) \frac{\partial p_l}{\partial x}. \quad (6)$$

By taking into account Eq. (4), from Eq. (6) we obtain

$$q_l = (1 - \varepsilon_s) \left[\frac{k}{\mu} \left(1 + \lambda_{pl} \frac{\partial}{\partial t} \right) \frac{\partial p_l}{\partial x} \right]_{x=0} - \varepsilon_s \left[\frac{k}{\mu} \left(1 + \lambda_{pl} \frac{\partial}{\partial t} \right) \frac{\partial p_l}{\partial x} \right]. \quad (7)$$

To derive a filtration equation, we use the following balance equation

$$\frac{\partial \varepsilon}{\partial t} = -\frac{\partial q_l}{\partial x}. \quad (8)$$

After substitution Eq. (7) into Eq. (8) and with taking into account $\varepsilon = 1 - \varepsilon_s$ we have

$$-\frac{\partial \varepsilon_s}{\partial t} = -\frac{\partial}{\partial x} \left\{ -\varepsilon_s \frac{k}{\mu} \left(1 + \lambda_{pl} \frac{\partial}{\partial t} \right) \frac{\partial p_l}{\partial x} - (1 - \varepsilon_s) \left[\frac{k}{\mu} \left(1 + \lambda_{pl} \frac{\partial}{\partial t} \right) \frac{\partial p_l}{\partial x} \right]_{x=0} \right\}$$

or

$$-\frac{\partial \varepsilon_s}{\partial t} = \frac{\partial}{\partial x} \left\{ -\varepsilon_s \frac{k}{\mu} \left(1 + \lambda_{pl} \frac{\partial}{\partial t} \right) \frac{\partial p_l}{\partial p_s} \frac{\partial p_s}{\partial x} \right\} - q_{lm} \frac{\partial \varepsilon_s}{\partial x}, \quad (9)$$

where

$$q_{lm} = -\frac{k}{\mu} \left(1 + \lambda_{pl} \frac{\partial}{\partial t} \right) \frac{\partial p_l}{\partial x} \Big|_{x=0}.$$

There are different relations between p_l and p_s . Some of the simplest possible relations are [5]:

- 1) Type 1: $dp_l + dp_s = 0$;
- 2) Type 2: $(1 - \varepsilon_s)dp_l + dp_s = 0$;
- 3) Type 3: $(1 - \varepsilon_s)dp_l + \varepsilon_s dp_s = 0$;
- 4) Type 4: $d[(1 - \varepsilon_s)dp_l] + d[\varepsilon_s dp_s] = 0$.

The general representation of these results can be written as [5]; $\frac{\partial p_l}{\partial p_s} = f'$, where f' for given

above types is:

- 1) Type 1: $f' = -1$;
- 2) Type 2: $f' = -\frac{1}{1 - \varepsilon_s}$;
- 3) Type 3: $f' = -\frac{\varepsilon_s}{1 - \varepsilon_s}$;
- 4) Type 4: $f' = -\frac{(1 - \varepsilon_s)p_0 - p_s}{(1 - \varepsilon_s)^2} \frac{d\varepsilon_s}{dp_s} - \frac{\varepsilon_s}{1 - \varepsilon_s}$.

So, from Eq. (9) we have

$$-\frac{\partial \varepsilon_s}{\partial t} = \frac{\partial}{\partial x} \left\{ -\varepsilon_s \frac{k}{\mu} \left(1 + \lambda_{pl} \frac{\partial}{\partial t} \right) f' \frac{\partial p_s}{\partial x} \right\} - q_{lm} \frac{\partial \varepsilon_s}{\partial x}. \quad (10)$$

If we use the relations [5] $\varepsilon_s = \varepsilon_s^0 \left(1 + \frac{p_s}{p_l} \right)^\beta$ and $k = k^0 \left(1 + \frac{p_s}{p_l} \right)^{-\delta}$ from Eq. (10) we obtain the following equation with respect to p_s

$$\frac{\varepsilon_s^0 \beta}{p_A} \left(1 + \frac{p_s}{p_l} \right)^{\beta-1} \frac{\partial p_s}{\partial t} = -\frac{k^0 \varepsilon_s^0}{\mu} \frac{\partial}{\partial x} \left[\left(1 + \frac{p_s}{p_l} \right)^{\beta-\delta} \left(1 + \lambda_{pl} \frac{\partial}{\partial t} \right) \left(f' \frac{\partial p_s}{\partial x} \right) \right] - \frac{\varepsilon_s^0 \beta}{p_A} q_{lm} \left(1 + \frac{p_s}{p_l} \right)^{\beta-1} \frac{\partial p_s}{\partial t}, \quad (11)$$

where ε_s^0 and k^0 denote respectively the values of value of ε_s and k at $p_s = 0$, $p_A = 0$ - characteristic pressure, β , δ signifies the compression effect due to p_s .

In the case $f' = -1$ (type 1) from Eq. (11) we have

$$\frac{\varepsilon_s^0 \beta}{p_A} \left(1 + \frac{p_s}{p_l} \right)^{\beta-1} \frac{\partial p_s}{\partial t} = \frac{k^0 \varepsilon_s^0}{\mu} \frac{\partial}{\partial x} \left[\left(1 + \frac{p_s}{p_l} \right)^{\beta-\delta} \left(1 + \lambda_{pl} \frac{\partial}{\partial t} \right) \frac{\partial p_s}{\partial x} \right] - \frac{\varepsilon_s^0 \beta}{p_A} q_{lm} \left(1 + \frac{p_s}{p_l} \right)^{\beta-1} \frac{\partial p_s}{\partial t}, \quad (12)$$

where expression for q_{lm} with respect to p_s is

$$q_{lm} = \frac{k}{\mu} \left(1 + \lambda_{pl} \frac{\partial}{\partial t} \right) \frac{\partial p_s}{\partial x} \Big|_{x=0}.$$

Eq. (12) is the basic equation of non-equilibrium filtration with cake formation. It can be solved with corresponding initial and boundary conditions, specified in particular on the moving boundary $x = L(t)$, which must be determined from the additional equation.

The initial and boundary conditions for Eq. (12) can be taken in the form

$$p_s(0, x) = 0, p_s(t, L(t)) = 0, -k \left(1 + \lambda_{pl} \frac{\partial}{\partial t} \right) \frac{\partial p_s}{\partial x} \Big|_{x=0} = \frac{p_0 - p_s}{R_m} \Big|_{x=0}, \quad (13)$$

where p_0 - suspension feed pressure.

2.2 Derivation of Equation for The Cake Growth

Now we proceed to the derivation of an equation for the mobile boundary $L(t)$, which characterizes the growth of the cake. The mobile boundary $L(t)$ expresses the thickness of the cake and continuously grows during the filtration process. The mobile boundary is determined on the basis of the mass conservation law on the surface of the cake, i.e. on the common boundary of the suspension and the cake. At $x < L(t)$ we have the cake, and at $x = L(t)$ we have the suspension. Then, the basic equation characterizing the growth of the cake can be taken in the form [5,6]

$$\frac{dL}{dt} = \frac{-q_l|_{L^+} - (-q_l|_{L^-})}{\varepsilon_{s_0} - \varepsilon_s^0}, \quad (14)$$

where ε_s^0 is solid content at zero pressure, ε_{s_0} is concentration of solid particles in suspension, $q_l|_{L^+}$ is filtration velocity at $x = L^+$, $q_l|_{L^-}$ is filtration velocity at $x = L^-$.

For the relaxation filtration law, we have the following relationships

$$q_l|_{L^+} + q_s|_{L^+} = q_l|_{L^-} + q_s|_{L^-} = q_{lm} = \left[-\frac{k}{\mu} \left(1 + \lambda_{pl} \frac{\partial}{\partial t} \right) \frac{\partial p_l}{\partial x} \right]_{x=0}. \quad (15)$$

The Darcy's law at $x = L^-$ can be written as

$$q_l|_{L^-} - \frac{1 - \varepsilon_s^0}{\varepsilon_s^0} q_s|_{L^-} = \left[-\frac{k}{\mu} \left(1 + \lambda_{pl} \frac{\partial}{\partial t} \right) \frac{\partial p_l}{\partial x} \right]_{x=L^-}. \quad (16)$$

From Eq. (15) we obtain

$$q_l|_{L^-} = -q_s|_{L^-} - \left[\frac{k}{\mu} \left(1 + \lambda_{pl} \frac{\partial}{\partial t} \right) \frac{\partial p_l}{\partial x} \right]_{x=0}. \quad (17)$$

Eq. (16) and Eq. (17) give

$$q_l|_{L^-} = -\varepsilon_s^0 \left[\frac{k}{\mu} \left(1 + \lambda_{pl} \frac{\partial}{\partial t} \right) \frac{\partial p_l}{\partial x} \right]_{x=L^-} - (1 - \varepsilon_s^0) \left[\frac{k}{\mu} \left(1 + \lambda_{pl} \frac{\partial}{\partial t} \right) \frac{\partial p_l}{\partial x} \right]_{x=0}. \quad (18)$$

In the suspension zone $x = L^+$ particles and liquid move with equal velocities

$$\frac{q_s|_{L^+}}{\varepsilon_{s_0}} = \frac{q_l|_{L^+}}{1 - \varepsilon_{s_0}},$$

that gives

$$q_s|_{L^+} = \frac{\varepsilon_{s0}}{1-\varepsilon_{s0}} q_l|_{L^+}, \quad (19)$$

By substitution Eq. (19) into Eq. (15) we obtain

$$q_l|_{L^+} = -(1 - \varepsilon_{s0}) \left[\frac{k}{\mu} \left(1 + \lambda_{pl} \frac{\partial}{\partial t} \right) \frac{\partial p_l}{\partial x} \right]_{x=0}. \quad (20)$$

With taking into account Eq. (18) and Eq. (20), from the Eq. (14) we have

$$\frac{dL}{dt} = \frac{\varepsilon_s^0}{\varepsilon_s^0 - \varepsilon_{s0}} \left[\frac{k}{\mu} \left(1 + \lambda_{pl} \frac{\partial}{\partial t} \right) \frac{\partial p_l}{\partial x} \right]_{L^-} + q_{lm}. \quad (21)$$

If we use the first type of the relationship between p_l and p_s , i.e. $f' = -1$, the Eq. (21) can be written with respect to p_s

$$\frac{dL}{dt} = -\frac{\varepsilon_s^0}{\varepsilon_s^0 - \varepsilon_{s0}} \left[\frac{k}{\mu} \left(1 + \lambda_{pl} \frac{\partial}{\partial t} \right) \frac{\partial p_s}{\partial x} \right]_{L^-} + q_{lm}. \quad (22)$$

By integrating the Eq. (22) the mobile front $L(t)$ can be determined. This equation is solved together with the basic equation of filtration Eq. (12) with Eq. (13).

We should note, that without relaxation effects, the analogous problems of filtration were considered in [5,9,10].

3. Numerical Analysis

Eq. (12), we write in the following form

$$\frac{\partial p_s}{\partial t} = \frac{k^0 p_A}{\beta \mu} \left(1 + \frac{p_s}{p_l} \right)^{1-\beta} \frac{\partial}{\partial x} \left[\left(1 + \frac{p_s}{p_l} \right)^{\beta-\delta} \left(1 + \lambda_{pl} \frac{\partial}{\partial t} \right) \frac{\partial p_s}{\partial x} \right] - q_{lm} \frac{\partial p_s}{\partial t}, \quad (23)$$

We introduce the following notations

$$a(p) = \left(1 + \frac{p_s}{p_l} \right)^{1-\beta}, \quad b(p) = \left(1 + \frac{p_s}{p_l} \right)^{\beta-\delta}, \quad c_2 = \frac{\varepsilon_s^0}{\varepsilon_s^0 - \varepsilon_{s0}}, \quad c(p) = \frac{k^0}{\mu} \left(1 + \frac{p_s}{p_l} \right)^{-\delta}, \quad c_1 = \frac{k^0 p_A}{\beta \mu}, \quad c^0(p) = \frac{k^0}{\mu} \left(1 + \frac{p_s}{p_l} \right)^{-\delta} \Big|_{x=0}.$$

With taking into account these notations, Eq. (23) can be transformed into the following form

$$\frac{\partial p_s}{\partial t} = c_1 a(p) \frac{\partial}{\partial x} \left[b(p) \left(1 + \lambda_{pl} \frac{\partial}{\partial t} \right) \frac{\partial p_s}{\partial x} \right] - q_{lm} \frac{\partial p_s}{\partial t}. \quad (24)$$

The equation for the mobile boundary $L(t)$, Eq. (22), takes the form

$$\frac{dL}{dt} = -c_2 \left[c(p) \left(1 + \lambda_{pl} \frac{\partial}{\partial t} \right) \frac{\partial p_s}{\partial x} \right]_{L^-} + q_{lm}, \quad (25)$$

where

$$q_{lm} = c^0(p) \left(1 + \lambda_{pl} \frac{\partial}{\partial t} \right) \frac{\partial p_s}{\partial x} \Big|_{x=0}.$$

To solve the problem Eq. (24), Eq. (25) with Eq. (13) and $L(0) = 0$, we use the finite differences method. We introduce a uniform grid by t with the step τ $\bar{\omega}_\tau = \{t | t = t_j = j\tau, j = 0, 1, \dots, N, \tau N = T\}$, and a non - uniform grid by coordinate x [36,37]

$$\bar{\omega}_h = \{x | x = x_i = x_{i-1} + h_i, i = 0, 1, \dots, N, N + 1, \dots, x_N = L\}$$

with the variable step $h_i > 0$.

We are to choose the step h_i from the section $[x_i, x_{i+1}]$ so, that the mobile boundary moves exactly on one step along the time grid. This approach is known as the method of catching the front in a grid node. We denote by ϕ_i^{j+1} the grid function corresponding to p_s . We approximate Eq. (24) by an implicit difference scheme that is nonlinear with respect to the function ϕ_i^{j+1}

$$\begin{aligned} \frac{\phi_i^{j+1} - \phi_i^j}{\tau} = c_1 \frac{2a(\phi_i^j)}{h_i + h_{i+1}} \frac{\partial}{\partial x} \left\{ b \left(\phi_{i+\frac{1}{2}}^{j+1} \right) \frac{\phi_{i+1}^{j+1} + \phi_{i-1}^{j+1}}{h_i + h_{i+1}} + \frac{\lambda_{pl}}{\tau} b \left(\phi_{i+\frac{1}{2}}^{j+1} \right) \left[\frac{\phi_{i+1}^{j+1} + \phi_{i-1}^{j+1}}{h_i + h_{i+1}} - \right. \right. \\ \left. \left. \frac{\phi_{i+1}^j + \phi_{i-1}^j}{h_i + h_{i+1}} \right] - b \left(\phi_{i-\frac{1}{2}}^{j+1} \right) \frac{\phi_i^{j+1} + \phi_i^{j+1}}{h_i} - \frac{\lambda_{pl}}{\tau} b \left(\phi_{i-\frac{1}{2}}^{j+1} \right) \left[\frac{\phi_i^{j+1} + \phi_{i-1}^{j+1}}{h_i} - \frac{\phi_i^j + \phi_{i-1}^j}{h_i} \right] \right\} - (q_{lm})_0^{j+1} \frac{\phi_i^{j+1} + \phi_{i-1}^{j+1}}{h_i}, \end{aligned} \quad (26)$$

where

$$\begin{aligned} a(\phi_i^j) = \left(1 + \frac{\phi_i^j}{p_A} \right)^{1-\beta}, \quad b \left(\phi_{i+\frac{1}{2}}^j \right) = \frac{1}{2} \left[\left(1 + \frac{\phi_{i+1}^j}{p_A} \right)^{\beta-\delta} + \left(1 + \frac{\phi_i^j}{p_A} \right)^{\beta-\delta} \right], \\ (q_{lm})_0^{j+1} = c^0(\phi_0^{j+1}) \left(\frac{\phi_1^{j+1} - \phi_0^{j+1}}{h_0} + \frac{\lambda_{pl}}{\tau} \left(\frac{\phi_1^{j+1} - \phi_0^{j+1}}{h_0} - \frac{\phi_1^j - \phi_0^j}{h_0} \right) \right), \quad c^0(\phi_0^{j+1}) = \frac{k^0}{\mu} \left(1 + \frac{\phi_0^{j+1}}{p_A} \right)^{-\delta}. \end{aligned}$$

Eq. (25) when $\frac{dL}{dt} \approx \frac{h_{i+1}}{\tau}$ after the approximation can be written in the form

$$\frac{h_{i+1}}{\tau} = -c_2 \left[c^0 \left(\phi_{i-\frac{1}{2}}^{j+1} \right) \left(\frac{\phi_i^{j+1} - \phi_{i-1}^{j+1}}{h_{i+1}} + \frac{\lambda_{pl}}{\tau} \left(\frac{\phi_i^{j+1} - \phi_{i-1}^{j+1}}{h_{i+1}} - \frac{\phi_i^j - \phi_{i-1}^j}{h_{i+1}} \right) \right) \right] + (q_{lm})_0^{j+1}, \quad (27)$$

where

$$c \left(\phi_{i-\frac{1}{2}}^j \right) = \frac{k^0}{2\mu} \left[\left(1 + \frac{\phi_i^j}{p_A} \right)^{-\delta} + \left(1 + \frac{\phi_{i-1}^j}{p_A} \right)^{-\delta} \right].$$

Approximation of initial and boundary conditions Eq. (13) gives

$$-\mu c^0(\phi_0^j) \left(\frac{\phi_1^{j+1} - \phi_0^{j+1}}{h_1} + \frac{1}{\tau} \left(\frac{\phi_1^{j+1} - \phi_0^{j+1}}{h_1} - \frac{\phi_1^j - \phi_0^j}{h_1} \right) \right) = \frac{p_0 + \phi_0^{j+1}}{R_m}, j = \overline{0, N},$$

$$\phi_i^{j+1} = 0, i = N + 1, N + 2, \dots, j = 0, 1, \dots \quad (28)$$

The obtained set of Eq. (26) is nonlinear, so to solve it we use the method of simple iteration

$$\frac{(\phi^{(s+1)})_i^{j+1} - \phi_i^j}{\tau} = c_1 \frac{2a(\phi_i^j)}{h_i + h_{i+1}} \frac{\partial}{\partial x} \left\{ b \left((\phi^{(s)})_{i+\frac{1}{2}}^{j+1} \right) \frac{(\phi^{(s+1)})_{i+1}^{j+1} + (\phi^{(s+1)})_{i-1}^{j+1}}{h_i + h_{i+1}} + \right.$$

$$\frac{\lambda_{pl}}{\tau} b \left((\phi^{(s)})_{i+\frac{1}{2}}^{j+1} \right) \left[\frac{(\phi^{(s+1)})_{i+1}^{j+1} + (\phi^{(s+1)})_{i-1}^{j+1}}{h_i + h_{i+1}} - \frac{\phi_{i+1}^j + \phi_{i-1}^j}{h_i + h_{i+1}} \right] - b \left((\phi^{(s)})_{i-\frac{1}{2}}^{j+1} \right) \frac{(\phi^{(s+1)})_i^{j+1} + (\phi^{(s+1)})_i^{j+1}}{h_i} -$$

$$\left. \frac{\lambda_{pl}}{\tau} b \left((\phi^{(s)})_{i-\frac{1}{2}}^{j+1} \right) \left[\frac{(\phi^{(s+1)})_i^{j+1} + (\phi^{(s+1)})_{i-1}^{j+1}}{h_i} - \frac{\phi_i^j + \phi_{i-1}^j}{h_i} \right] \right\} - (q_{lm})_0^{j+1} \frac{(\phi^{(s+1)})_i^{j+1} + (\phi^{(s+1)})_{i-1}^{j+1}}{h_i}, \quad (29)$$

where

$$b \left((\phi^{(s)})_{i+\frac{1}{2}}^j \right) = \frac{1}{2} \left[\left(1 + \frac{(\phi^{(s)})_{i+1}^{j+1}}{p_A} \right)^{\beta-\delta} + \left(1 + \frac{(\phi^{(s)})_i^{j+1}}{p_A} \right)^{\beta-\delta} \right],$$

$$(q_{lm})_0^{j+1} = c^0 \left((\phi^{(s)})_0^{j+1} \right) \left(\frac{(\phi^{(s)})_1^{j+1} - (\phi^{(s)})_0^{j+1}}{h_0} + \frac{\lambda_{pl}}{\tau} \left(\frac{(\phi^{(s)})_1^{j+1} - (\phi^{(s)})_0^{j+1}}{h_0} - \frac{\phi_1^j - \phi_0^j}{h_0} \right) \right),$$

s is the number of iteration.

It can be seen that the system of Eq. (29) is now linear with respect to $(\phi^{(s+1)})_i^{j+1}$, which allows us to use the Tomas' algorithm. As a condition to stop iteration procedure on this time layer, the following relationship can be used:

$$\max_i \left| (\phi^{(s+1)})_i^{j+1} - (\phi^{(s)})_i^{j+1} \right| \leq \Lambda, \quad (30)$$

where Λ is a small parameter.

When condition (30) is satisfied then $(\phi^{(s+1)})_i^{j+1} = \phi_i^{j+1}$. As an initial approach we can use $(\phi^{(s=0)})_i^{j+1} = \phi_i^j$.

Eq. (29) leads to the system of linear algebraic equations

$$A_i^{(s)} (\phi^{(s+1)})_{i-1}^{j+1} - B_i^{(s)} (\phi^{(s+1)})_i^{j+1} + C_i^{(s)} (\phi^{(s+1)})_{i+1}^{j+1} = -F_i^{(s)}, i = \overline{1, N-1}, \quad (31)$$

where

$$A_i^{(s)} = -\frac{1}{h_i + h_{i+1}} \left(1 + \frac{\lambda_{pl}}{\tau} \right) b \left((\phi^{(s)})_{i+\frac{1}{2}}^{j+1} \right) + \frac{1}{h_i} \left(1 + \frac{\lambda_{pl}}{\tau} \right) b \left((\phi^{(s)})_{i-\frac{1}{2}}^{j+1} \right) + \frac{h_i + h_{i+1}}{2c_1 h_i a(\phi_i^j)} q_{lm},$$

$$\begin{aligned}
 B_i^{(s)} &= \frac{1}{h_i} \left(1 + \frac{\lambda_{pl}}{\tau} \right) b \left((\phi^{(s)})_{i-\frac{1}{2}}^{j+1} \right) + \frac{h_i+h_{i+1}}{2\tau c_1 a(\phi_i^j)} + \frac{h_i+h_{i+1}}{2c_1 h_i a(\phi_i^j)} q_{lm}, \\
 C_i^{(s)} &= \frac{1}{h_i+h_{i+1}} \left(1 + \frac{\lambda_{pl}}{\tau} \right) b \left((\phi^{(s)})_{i+\frac{1}{2}}^{j+1} \right), \\
 F_i^{(s)} &= \frac{h_i+h_{i+1}}{2\tau c_1 a(\phi_i^j)} \phi_i^j + \frac{\lambda_{pl}}{(h_i+h_{i+1})\tau} b \left((\phi^{(s)})_{i+\frac{1}{2}}^{j+1} \right) (\phi_{i-1}^j - \phi_{i+1}^j) - \frac{\lambda_{pl}}{h_i \tau} b \left((\phi^{(s)})_{i-\frac{1}{2}}^{j+1} \right) (\phi_{i-1}^j - \phi_i^j).
 \end{aligned}$$

The Eq. (27) is used to determine the step h_{i+1} and it can be written in the form

$$(h_{i+1})^2 - \tau(q_{lm})_0^{j+1} h_{i+1} + \tau c_2 c \left(\phi_{i-\frac{1}{2}}^j \right) \left(\phi_i^{j+1} - \phi_{i-1}^{j+1} + \frac{\lambda_{pl}}{\tau} (\phi_i^{j+1} - \phi_{i-1}^{j+1} - \phi_i^j + \phi_{i-1}^j) \right) = 0 \quad (32)$$

By solving this nonlinear equation for each temporal layer, we can determine h_{i+1} . The system of linear algebraic Eq. (31) is solved by the Tomas' algorithm

$$(\phi^{s+1})_i^{j+1} = \alpha_{i+1} (\phi^{s+1})_{i+1}^{j+1} + \beta_{i+1}, \quad i = 0, 1, \dots, \quad (33)$$

where

$$\alpha_{i+1} = \frac{C_i^{(s)}}{B_i^{(s)} - A_i^{(s)} \alpha_i}, \quad \beta_{i+1} = \frac{F_i^{(s)} + A_i^{(s)} \beta_i}{B_i^{(s)} - A_i^{(s)} \alpha_i}.$$

The starting values of the coefficients α_1 and β_1 are determined from the boundary condition (33), which have the form

$$\alpha_1 = \frac{R_m \mu c^0 (\phi_0^j) \left(1 + \frac{\lambda_{pl}}{\tau} \right)}{R_m \mu c^0 (\phi_0^j) \left(1 + \frac{\lambda_{pl}}{\tau} \right) + h_0}, \quad \beta_1 = \frac{R_m \mu c^0 (\phi_0^j) (\phi_0^j - \phi_1^j) + h_0 p_0}{R_m \mu c^0 (\phi_0^j) \left(1 + \frac{\lambda_{pl}}{\tau} \right) + h_0}.$$

4. Numerical Results

Numerical results with using Eq. (31), Eq. (32) were obtained with the following values of the parameters: $p_A = 10^4 \text{ Pa}$, $p_0 = 10^5 \text{ Pa}$, $R_m = 10^{12} \text{ 1/m}$, $\mu = 10^{-3} \text{ Pa}\cdot\text{s}$, $k^0 = 10^{-13} \text{ m}^2$, $\varepsilon_s^0 = 0.20$, $\varepsilon_{s_0} = 0.0076$, $\beta = 0.13$, $\delta = 0.57$. These values are typical for cake filtration process [5,6].

Some results are graphically shown below. The dynamics of the compression pressure and the pressure in the liquid phase at a given point x for different values of the relaxation time λ_{pl} is shown in Figure 2(a). At constant values of the all other parameters increasing in the relaxation time leads to a slow grows of the compression pressure, i.e. in the dynamics of the pressure one can see the lag. This phenomenon effects on all other filtration characteristics.

On the contrary, in pressure for the liquid phase relaxation effects lead to the advanced dynamics (Figure 2(b)). Within the framework of assumptions made above we have $p_0 = p_l + p_s$. Comparing two figures, Figure 2(a) and Figure 2(b), we can check the implementation of the $p_0 = p_l + p_s$. By summing values p_l and p_s from corresponding curves of Figure 2(a) and Figure 2(b) we obtain p_0 . In addition, the fulfilment of the condition $p_0 = p_l + p_s$ shows us correctness of numerical computations.

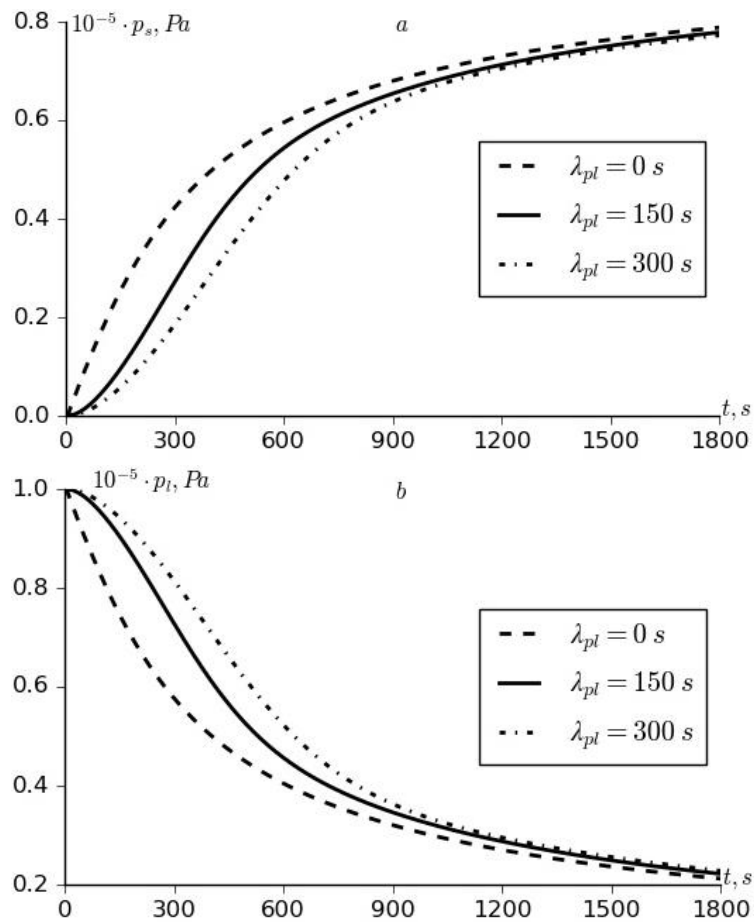
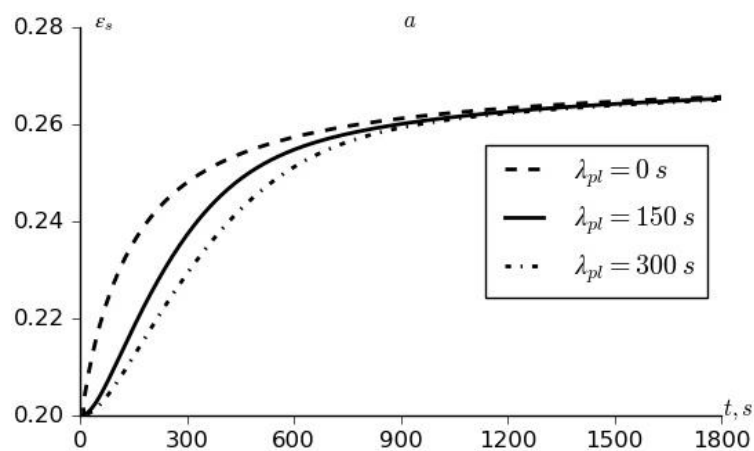


Fig. 2. Dynamics of p_s (a), p_l (b) in $x = 0.02$ m

Dynamics of ε_s and k/k_0 at a given point of the cake is also determined (Figure 3). The curves indicate that the relaxation phenomena lead to slow dynamics of ε_s and faster dynamics of k/k_0 . In other words, relaxation phenomena in filtration law cause slow dynamics of the volume fraction of particles, i.e. solidosity. As a result, relative permeability of the cake is greater at a given point and given time. So, one can conclude that relaxation phenomena increase fluid relative velocity through cake for a given pressure p_0 and, consequently, for a given pressure gradient.



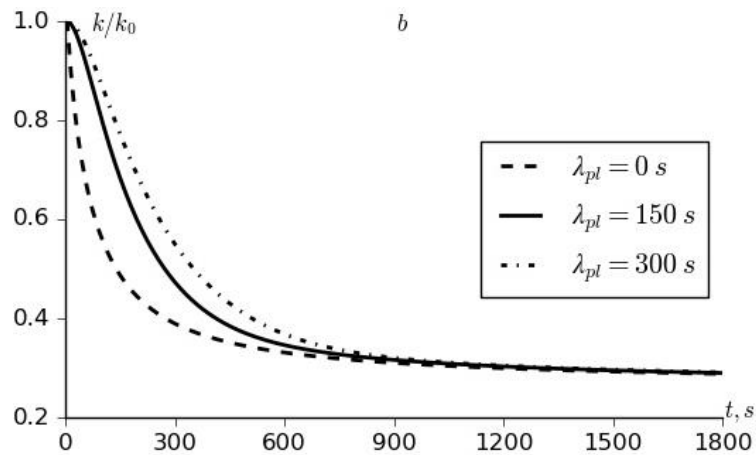


Fig. 3. Dynamics of ε_s (a), k/k_0 (b) in $x = 0.02$ m.

The growth of the cake thickness for different values of the relaxation time λ_{pl} is shown in Figure 4. As one can see the increasing of relaxation time λ_{pl} leads to the faster growth of the cake thickness at all other constant conditions. In other words, relaxation effects cause more intensive transfer of particles from suspension to the cake. It, in turn, alters all other filtration characteristics, such as fluid pressure distribution, compressive pressure, porosity and consolidation of the cake, fluid flow rate through the cake and effective hydrodynamic resistance.

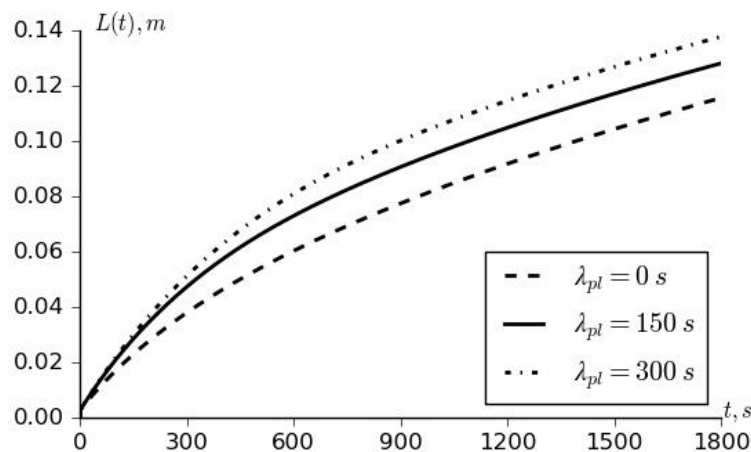


Fig. 4. Dynamics of the cake thickness

Dynamics of q_{lm} at different relaxation time λ_{pl} is shown in Figure 5(a). As we see from Figure 5(a) with relaxation phenomena q_{lm} has more intensive dynamics for a same time period, duration of which is characterized by the value of λ_{pl} . Outside of this time period values of q_{lm} for different λ_{pl} are close, that indicates the ending of the influence of λ_{pl} . So, transient relaxation period is completed.

With using curves of Figure 5(a) total filtrate flow rate Q_{lm} is determined as

$$Q_{lm} = \int_0^t q_{lm} dt.$$

Some representative curves of Q_{lm} are shown in Figure 5(b). They show that for greater relaxation times λ_{pl} , the total mass of filtered liquid for a given time is greater.

All analyzed results show that relaxation phenomena cause considerable change of filtration characteristics.

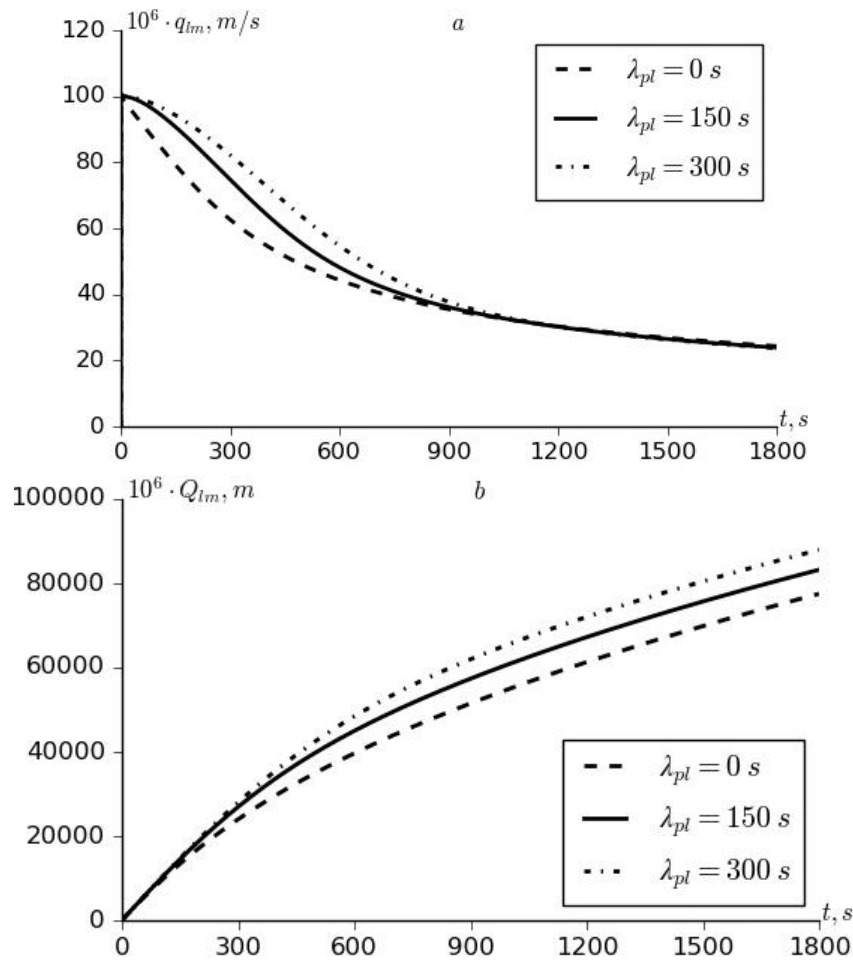


Fig. 5. Dynamics of q_{lm} (a), Q_{lm} (b)

5. Conclusions

In the paper, relaxation cake filtration equations are derived. Filtration law for liquid/particle relative velocity is written in differential relaxation form with triple relaxations, one - with respect to superficial liquid velocity q_l , the second one-with respect to the superficial particle velocity q_s , the third one-with respect to liquid pressure gradient $\partial p_l / \partial x$. There the only last case was analysed. Using power law dependences of solidosity ε_s and permeability k of the cake with respect to compressive pressure p_s , as well as relations between p_l and p_s a relaxation filtration equation is derived, that is written with respect to p_s . Using balance law on the common boundary of suspension and cake, a relaxation equation for cake thickness $L(t)$ is also derived.

Equations are numerically solved. Influence of the relaxation parameter λ_{pl} on filtration characteristics is established. It is shown that relaxation effects lead to the slow dynamics of p_s and more intensive dynamics of p_l at a given point. Similarly, relaxation phenomena lead to slow dynamics of ε_s and faster dynamics of k/k_0 . So, relaxation phenomena in filtration law cause slow dynamics of the solidosity and faster dynamics of relative permeability. All of obtained results indicate that relaxation phenomena in filtration law increase fluid relative velocity through the cake for a given inlet pressure p_0 and, as a consequence, for a given pressure gradient.

It was established that the increasing of the relaxation time leads to the faster growth of the cake thickness, i.e. relaxation phenomena cause more intensive sedimentation of the particles from suspension to the cake. Relaxation phenomena alter also other filtration characteristics such as fluid pressure distribution, compressive pressure, porosity and consolidation of the cake, fluid flow rate through the cake effective hydrodynamic resistance etc.

Dynamics of q_{lm} and total filtrate flow rate Q_{lm} also analysed for different relaxation times. It was shown that with increasing of relaxation phenomena dynamics of q_{lm} becomes more intensive. This dependence we can observe in dynamics of Q_{lm} also.

On the basis of obtained results one can conclude that relaxation phenomena cause considerable change of filtration characteristics.

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