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Finite Difference Solution for MHD Flow Between Two Parallel Permeable Plates with Velocity Slip

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ARTICLE INFO	ABSTRACT
	The problem of Hydromagnetic steady laminar flow of an electrically conducting viscous incompressible fluid between parallel pates has been studied. The nonzero tangential slip velocity at the permeable boundary is considered. A numerical solution is derived for the governing nonlinear boundary value problem using a novel Keller box scheme. The effect of suction Reynold number R, Hartman number M and slip coefficient ϕ on derived quantities such as velocity filed and skin friction at the boundaries are analyzed. The physical significance of the flow parameters is also discussed.

1. Introduction

The study of Magentohydrodynamics (MHD) flow of an incompressible, viscous and electrically conducting fluid between two porous plates not only possesses the theoretical appeal but also has many real-world engineering applications such as magentohydrodynamic pumps, geothermal energy extraction, MHD generators [1]. A useful survey of MHD theory and its applications can be found in the book by Moreau [2]. Owing to its important applications, researchers have used this phenomenon to study fluid flow behavior under various geometries, like vertical plates, stretching sheets, channels, tubes, and disks. Hartmann [3] investigated the impact of a transverse uniform magnetic field on the steady flow of an electrically conducting fluid between two parallel plates.

The problem of homogeneous fluid flowing between porous walls with uniform suction at the wall was first investigated by Berman [4] and obtained the perturbation solution for the velocity field for small suction Reynold number. Later, Berman's problem was extended by many researchers for both small and large Reynold number [5-12]. A distinctive evaluation of forced convection heat transfer to an electrically conducting liquid flowing in a channel with a transverse magnetic field was

carried by Perlmutter and Siegel [13]. Subsequently, Attia and Kotb [14] and Attia [15] investigated the effect of temperature based viscosity on the magentohydrodynamic flow in a channel. The steady MHD flow and heat transfer induced by an exponentially shrinking sheet with partial slip, thermal radiation and suction was investigated using MATLAB bvp4c solver by Adnan *et al.,* [16]. MHD slip Darcy flow of viscoelastic fluid over a stretching surface in a porous medium with thermal radiation and viscous dissipation was analytically examined by Wahid *et al.,* [17].

Most of the above analysis are confined to no-slip boundary conditions. But the velocity slip being present at the permeable boundaries has a remarkable effect on the fluid flow [18-21]. In the present study, the problem of magnetohydrodynamic flow of an incompressible viscous fluid flowing between two parallel porous plates is investigated in the presence of external magnetic field and velocity slip boundary conditions at the permeable boundaries. The numerical solution of the governing problem is obtained and is analyzed for all flow parameters using the Keller-box method. Keller [22] used this scheme for solving diffusion problems, but it has subsequently been applied to a broad class of problems. A sketch of applications of this box scheme to a variety of boundary layer flow problems is given in researches by Keller [23] and Vajravelu and Prasad [24].

An outline of the rest of this paper is as follows. In section (2) mathematical formulation of the proposed problem is explained. Section (3) devotes to obtain a numerical solution of the governing problem and section (4) presents results and discussions. Important conclusions from the study are given in section (5).

2. Mathematical Formulation

The steady laminar flow of an incompressible viscous fluid between two parallel porous plates is considered with slip at the porous boundaries. An external magnetic field of strength H_0 is applied in the direction perpendicular to the walls. Assuming a $XY-$ coordinate system with the origin at center of the channel, L be the length of the channel with $2l$ being the distance between two plates as shown in Figure 1. Let u and v be the velocity components in the x and y directions, respectively.
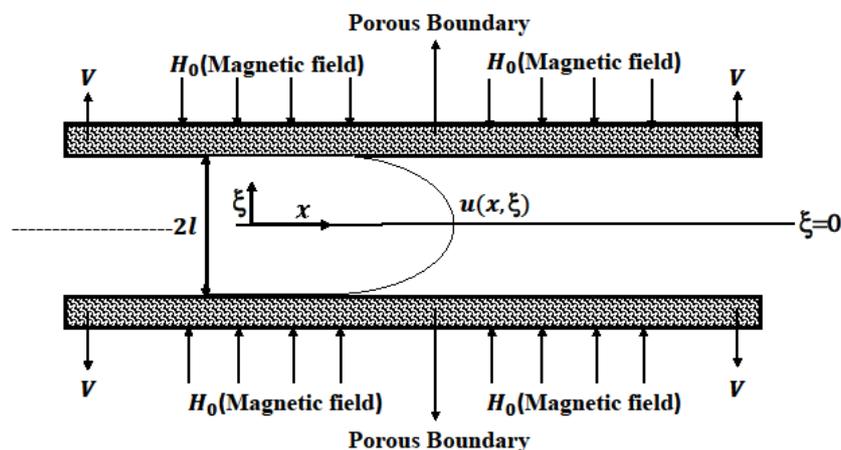
Fig. 1. Geometry of viscous fluid flow between porous plates

The governing equations of motion along with continuity equation are

$$\frac{\partial u}{\partial x} + \frac{1}{l}\frac{\partial v}{\partial \xi} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + \frac{v}{l}\frac{\partial u}{\partial \xi} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + v\left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{l^2}\frac{\partial^2 u}{\partial \xi^2}\right) - \frac{\sigma B^2 u}{\rho} \tag{2}$$

$$u\frac{\partial v}{\partial x} + \frac{v}{l}\frac{\partial v}{\partial \xi} = -\frac{1}{\rho l}\frac{\partial p}{\partial \xi} + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{1}{l^2}\frac{\partial^2 v}{\partial \xi^2}\right) \tag{3}$$

where, σ is the electrical conductivity and $B = \mu_e H_0$, μ_e is the magnetic permeability. Also, $\xi = \frac{y}{l}$ is the dimensionless variable, ρ is the density, p is the pressure. The appropriate boundary conditions are;

$$u(x, \pm 1) = \pm u_{\text{slip}}, \tag{4}$$

$$v(x, \pm 1) = \pm V \tag{5}$$

where $u_{\text{slip}} = -\frac{\sqrt{k}}{\alpha l}\frac{\partial u}{\partial \eta}$ is the slip velocity and V is the suction velocity at walls of the channel. By introducing the stream function of the form [4]

$$\psi(x, y) = (lU(0) - Vx)F(\xi) \tag{6}$$

where $U(0)$ is the average entrance velocity at $x = 0$. The expressions for axial and radial components of velocity can be derived as

$$u = \frac{1}{l}(lU(0) - Vx)F'(\xi) \tag{7}$$

$$v = VF(\xi) \tag{8}$$

Substituting Eq. (7) and Eq. (8) in Eq. (2) and Eq. (3) the momentum equations reduces to

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \left(U(0) - \frac{Vx}{l}\right)\left(\frac{V}{l}(FF'' - F'^2) - \frac{v}{l^2}F''' + \frac{\sigma B^2}{\rho}F'\right) - \frac{1}{lp}\frac{\partial p}{\partial \xi} = \frac{V^2}{l}FF' - \frac{vV}{l^2}F''$$

Eliminating the pressure term and upon simplification, we have,

$$\frac{d}{d\xi}\left(\frac{V}{l}(FF'' - F'^2) - \frac{v}{l^2}F''' + \frac{\sigma B^2}{\rho}F'\right) = 0 \tag{9}$$

$$F''' + R(F'^2 - FF'') - aRF' = K \tag{10}$$

where $R = \frac{lV}{v}$ is suction Reynolds number, $a = \frac{H_0^2 \mu_e^2 \sigma l}{\rho V}$ and K is an arbitrary constant. Consider $aR = M^2$, with $M = Bl(\frac{\sigma}{vp})^{1/2}$ called Hartmann number. The boundary conditions can be rewritten as

$$F(1) = 1, \ F'(1) = -\phi F''(1), \ F(-1) = -1, \ F'(-1) = \phi F''(-1) \tag{11}$$

where $\phi = \frac{\sqrt{k}}{\alpha l}$ is the slip coefficient.

3. Finite Difference Solution

The fourth-order differential equation in Eq. (10) governing the flow along with the suitable boundary conditions in Eq. (11) is solved by a novel finite-difference technique developed by Keller also known as the 'box' method. This Keller box method is an implicit scheme with second-order accuracy in both space and time. This scheme differs from other techniques, where the second and higher-order derivatives are replaced by first derivatives through the introduction of additional variables, which results in a system of first-order equations. And then, these first-order derivatives are approximated using appropriate differences, which gives a system of difference equations. Using suitably the boundary conditions result in an algebraic block tri-diagonal system and can be solved with the general block tri-diagonal algorithm.

The method begins with introducing new variables to reduce the fourth order governing equation to four system of first order equations as follows.

$$U = F', \ V = F'', W = F''',$$
$$W' + R(UV - FW) - M^2 V = 0 \tag{12}$$

The corresponding boundary conditions can be written as

$$F(1) = 1, \qquad F(-1) = -1$$
$$U(1) = -\phi V(1), \quad U(-1) = \phi V(-1) \tag{13}$$

Following this step, the system of first order equations are discretized by substituting the appropriate finite difference approximations. Accordingly, the system from Eq. (12) can be written as

$$F_j - F_{j-1} = d_j(U_j + U_{j-1}) \tag{14}$$

$$U_j - U_{j-1} = d_j(V_j + V_{j-1}) \tag{15}$$

$$V_j - V_{j-1} = d_j(W_j + W_{j-1}) \tag{16}$$

$$W_j - W_{j-1} + \frac{Rd_j}{2}(U_j + U_{j-1})(V_j + V_{j-1}) - \frac{Rd_j}{2}(F_j + F_{j-1})(W_j + W_{j-1}) - d_j M^2 (V_j + V_{j-1}) = 0 \tag{17}$$

This nonlinear discretized system is linearized using Newton's linearisation method. Substituting Newton's iterations as, $()_j^{(i+1)} = ()_j^{(i)} + \delta()_j^{(i)}$ for all the variables in the system, we obtain the following linear set of algebraic equations.

$$\delta F_j - \delta F_{j-1} - d_j(\delta U_j + \delta U_{j-1}) = (r_1)_j \tag{18}$$

$$\delta U_j - \delta U_{j-1} - d_j(\delta V_j + \delta V_{j-1}) = (r_2)_j \tag{19}$$

$$\delta V_j - \delta V_{j-1} - d_j(\delta W_j + \delta W_{j-1}) = (r_3)_j \tag{20}$$

$$(a_1)_j \delta W_j + (a_2)_j \delta W_{j-1} + (a_3)_j(\delta V_j + \delta V_{j-1}) + (a_4)_j(\delta U_j + \delta U_{j-1}) = (r_4)_j \tag{21}$$

where,

$$(a_1)_j = 1 - \frac{d_j R}{2}(F_j + F_{j-1}) \tag{22}$$

$$(a_2)_j = -2 + (a_1)_j \tag{23}$$

$$(a_3)_j = \frac{R d_j}{2}(U_j + U_{j-1}) - M^2 d_j \tag{24}$$

$$(a_4)_j = \frac{-d_j R}{2}(W_j + W_{j-1}) \tag{25}$$

$$(a_5)_j = -\frac{d_j R}{2}(V_j + V_{j-1}) \tag{26}$$

$$(r_1)_j = d_j(U_j + U_{j-1}) + F_{j-1} - F_j \tag{27}$$

$$(r_2)_j = d_j(V_j + V_{j-1}) + U_{j-1} - U_j \tag{28}$$

$$(r_3)_j = d_j(W_j + W_{j-1}) + V_{j-1} - V_j \tag{29}$$

$$(r_4)_j = M^2 d_j(V_j + V_{j-1}) - \frac{R d_j}{2}((U_{j-1} + U_j)(V_{j-1} + V_j) - (F_{j-1} + F_j)(W_{j-1} + W_j)) + W_{j-1} - W_j \tag{30}$$

The linearized system of Eq. (18)-(21) have a block tri-diagonal structure which can be solved with the help of block elimination method. In vector -matrix form, Eq. (18)-(21) can be written as

$$A\delta = r \tag{31}$$

where,

$$A = \begin{bmatrix} [A_1] & [C_1] & & & & \\ & [A_2] & [C_2] & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & \ddots & \\ & & & [B_{J-1}] & [A_{J-1}] & [C_{J-1}] \\ & & & & [B_J] & [A_J] \end{bmatrix} \tag{32}$$

$$\delta = \begin{bmatrix} [\delta_1] \\ \vdots \\ \\ \end{bmatrix} \quad r = \begin{bmatrix} [r_1] \\ \vdots \\ \\ \end{bmatrix} \tag{33}$$

where in Eq. (32) the elements are defined by

$$[A_j] = \begin{bmatrix} 0 & 0 & 1 & -d_j \\ -d_j & 0 & 0 & 1 \\ -1 & -d_j & 0 & 0 \\ (a_3)_j & (a_2)_j & (a_4)_j & (a_5)_j \end{bmatrix} \text{ for } 1 \leq j \leq J-1; \quad [A_J] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -d_j & 0 & -d_j & 0 \\ -1 & -d_j & 1 & -d_j \\ (a_3)_J & (a_2)_J & (a_3)_J & (a_1)_J \end{bmatrix}$$

$$[A_J] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -d_j & 0 & -d_j & 0 \\ -1 & -d_j & 1 & -d_j \\ (a_3)_J & (a_2)_J & (a_3)_J & (a_1)_J \end{bmatrix}; \quad [B_j] = \begin{bmatrix} 0 & 0 & -1 & -d_j \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (a_4)_j & (a_5)_j \end{bmatrix}, \quad 2 \leq j \leq J$$

$$[C_j] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -d_j & 0 & 0 & 0 \\ 1 & -d_j & 0 & 0 \\ (a_3)_j & (a_1)_j & 0 & 0 \end{bmatrix}, \quad 1 \leq j \leq J-1$$

$$[\delta_j] = [\delta V_{j-1} \quad \delta W_{j-1} \quad \delta F_j \quad \delta U_j]^T, \quad 1 \leq j \leq J-1$$

and

$$[\delta_J] = [\delta V_{J-1} \quad \delta W_{J-1} \quad \delta V_J \quad \delta W_J]^T$$

$$[r_j] = [(r_1)_j \quad (r_2)_j \quad (r_3)_j \quad (r_4)_j]^T, \quad 1 \leq j \leq J$$

Now we write,

$$A = Lu \tag{34}$$

where,

$$L = \begin{bmatrix} [\alpha_1] & & & \\ & [\alpha_2] & & \\ & & \ddots & \\ & & & \ddots \\ & & & [B_J] \quad [\alpha_J] \end{bmatrix}, u = \begin{bmatrix} [I] & [\Gamma_1] & & & \\ & [I] & [\Gamma_2] & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & [I] & [\Gamma_{J-1}] \\ & & & & [I] \end{bmatrix}$$

Here, $[I]$ is unit matrix and $[\alpha_i]$ and $[\Gamma_i]$ are 4×4 matrices whose elements are determined by the following equations

$$[\alpha_j] = [A_1]$$

$$[A_1][\Gamma_1] = [C_1]$$

$$[\alpha_j] = [A_j] - [B_j][\Gamma_{j-1}], \quad j = 2,3,,J$$

$$[\alpha_j][\Gamma_j] = [C_j], \quad j = 2,3,,J-1$$

Eq. (34) substituted in Eq. (31) and we have

$$Lu\delta = r \tag{35}$$

Defining $u\delta = w$, we have

$$Lw = r \tag{36}$$

where,

$$w = [[w_1]\ [w_2]\ [w_3]\ [w_4]]^T \tag{37}$$

and the $[w_j]$ are 4×1 column matrices. The elements w can be solved from Eq. (36)

$$[\alpha_1][w_1] = [r_1]$$

$$[\alpha_j][w_j] = [r_j] - [B_j][w_{j-1}]$$

Once the elements of w are found, solution for δ can be obtained using relations:

$$[\delta_J] = [w_J]$$

$$[\delta_j] = [w_j] - [\Gamma_j][\delta_{j+1}]$$

Theses solution for δ can be used to find $(i + 1)^{th}$ iteration.

4. Results and Discussion

The numerical study on Magnetohydrodynamics (MHD) flow between parallel plates with velocity slip at the permeable boundaries is performed using a novel Keller-Box method. We obtain the solution for the derived quantities from the governing non-linear boundary value problem and are shown in the graphs. The velocity profiles, both axial as well as radial velocity are plotted in Figure 2 to Figure 5 for different positive values of flow Reynold number R, Hartmann Number M and slip coefficient ϕ. The presence of both the slip coefficient and magnetic parameter decreased the magnitude of both axial and radial velocity profiles in general. The plot also reveals that as magnetic field intensity M increases the axial velocity decreases near the central plane and fluid is pushed towards the boundaries. Moreover, as M increases velocity profiles show the characteristic flattering.
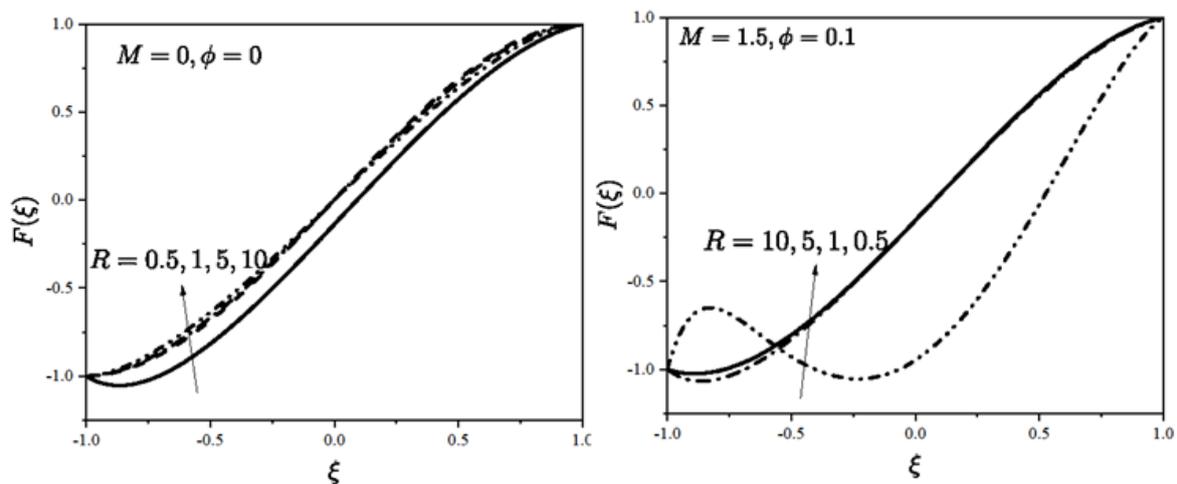
Fig. 2. Dimensionless Axial velocity for various Reynolds number with $M = 0,1.5$ and $\phi = 0,0.1$
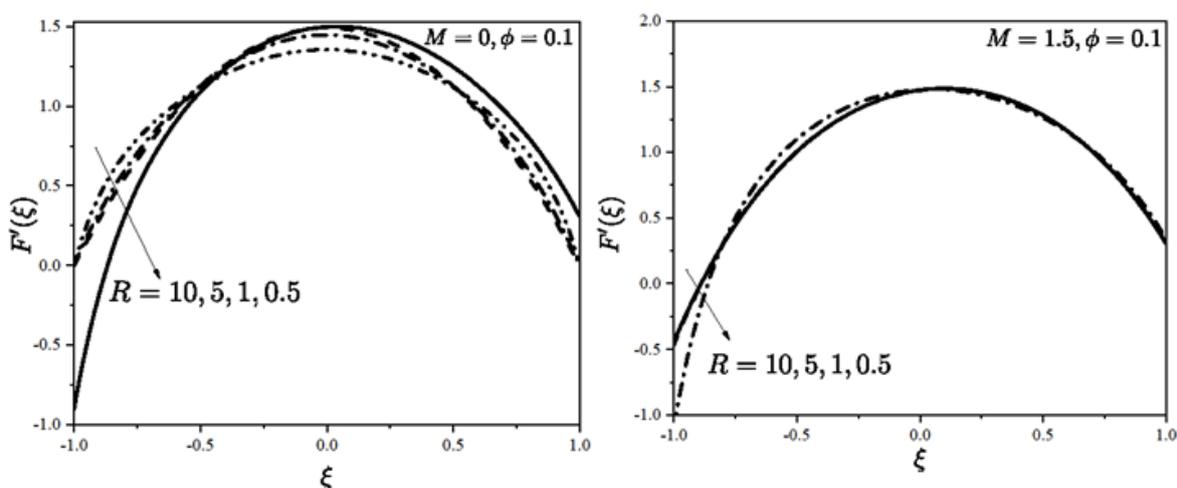
Fig. 3. Dimensionless radial velocity for various Reynolds number with $M = 0,1.5$ and $\phi = 0.1$
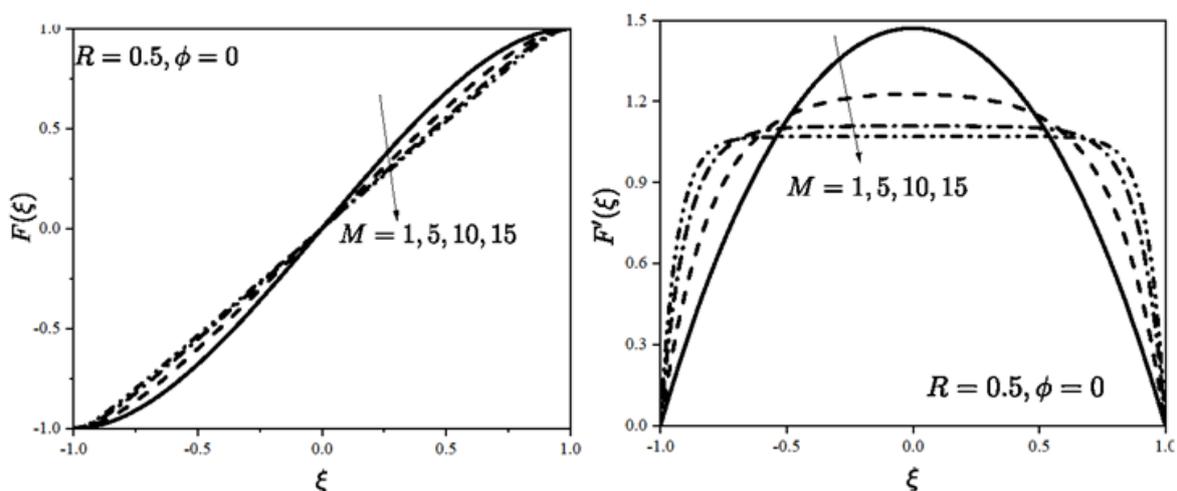
Fig. 4. Dimensionless Axial velocity and radial velocity for various M with $R = 0.5$ and $\phi = 0$

When $\phi = 0$(no slip case) and $M = 0$, the governing problem reduced to Berman's problem [4]. Moreover, it is evident from Figure 5 that increasing slip leads to flattening of profiles and reduced wall shear stress. A similar observation is made in Figure 2 for the positive value of flow Reynolds number R. Dimensionless radial velocity profiles depicted in Figure 3, which show the effect of slip

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coefficient ϕ for fixed suction and M. Irrespective of the values of pertinent parameters, the axial velocity profile takes its dimensionless value 1 at the upper plate and reduces to -1 at the plate and with a point of inflection near the central plane. From Figure 2, the negative axial velocity near the walls in the presence of slip proves the occurrence of flow reversal near the boundaries.
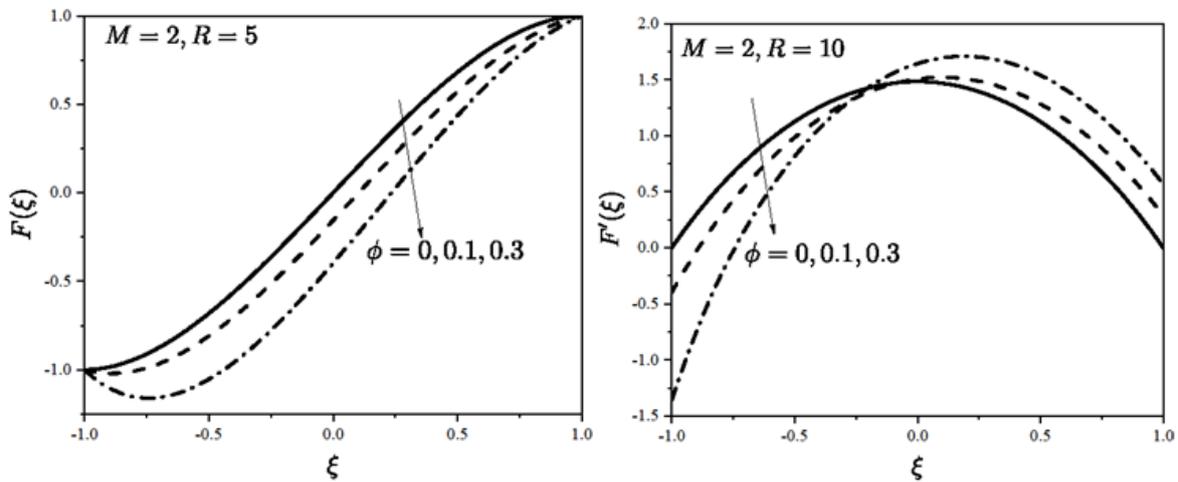
Fig. 5. Dimensionless Axial velocity and radial velocity for various slip coefficient ϕ with $M = 2$

The coefficient of skin friction near the plate $-F''(1)$ is computed in the Table 1. An increase in the magnitude of skin friction is seen with increasing suction. Meanwhile, a general decrease in the skin friction coefficient is observed with an increase in velocity slip.

Table 1
Calculation of skin friction coefficient ($-F''(1)$) for different values of magnetic parameter M, Reynolds number R and velocity slip parameter ϕ

M		$\phi = 0$	$\phi = 0.1$	$\phi = 0.2$
0	$R = 1$	3.0586	2.7935	2.7411
0.5		3.1122	2.8233	2.7709
1		3.2677	2.9091	2.8624
1.5		3.5099	3.0417	3.0274
0	$R = 5$	3.6936	2.7098	1.8262
0.5		3.7901	2.7941	1.7891
1		4.0586	2.9996	1.9125
1.5		4.4494	3.2631	2.1764

5. Conclusion

The present analysis focused on the finite difference solution to understand the combined effects of slip coefficient and magnetic field on the steady flow of viscous incompressible fluid flowing between two parallel plates with permeable boundaries. We infer that the fluid velocity is reduced by both the magnetic field and slip parameter. It is well known that the suction Reynolds number increases skin friction, the present study also reveals the effects of the magnetic field and slip velocity on the variation in skin friction at the porous boundaries.

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