The Effect of Different Scale on Object to the Approximation of the First Order Polarization Tensor of Sphere, Ellipsoid, and Cube

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ARTICLE INFO	ABSTRACT
	Polarization tensor (PT) is a classical terminology in fluid mechanics and theory of electricity that can describe geometry in a specific boundary domain with different conductivity contrasts. In this regard, the geometry may appear in a different size, and for easy characterizing, the usage of PT to identify particular objects is crucial. Hence, in this paper, the first order polarization tensor for different types of objects with a diverse range of sizes are presented. Here, we used three different geometries: sphere, ellipsoid, and cube, with fixed conductivity for each object. The software Matlab and Netgen Mesh Generator are the essential mathematical tools to aid the computation of the polarization tensor. From the analytical results obtained, the first order PT for sphere and ellipsoid depends on the size of both geometries. On the other hand, the numerical investigation is conducted for the first order PT for cube, since there is no analytical solution for the first order PT related to this geometry, to further verify the scaling property of the first order PT due to the scaling on the size of the original related object. Our results agree with the previous theoretical result that the first order polarization tensor of any geometry will be scaled at a fixed scaling factor according to the scaling on the size of the original geometry.

1. Introduction

Polarization tensor (PT) or just polarization is classically studied together with the virtual mass in fluid mechanics and hydrodynamics [1]. It was then extended and generalized by Ammari and Kang [2], where the researchers named this new PT as the Generalized Polarization Tensor (GPT). Ammari and Kang used the mathematical formulation of PT and applied it into a simulation on a cloaking device [3]. The first term of GPT is the first order PT [4,5]. The study of polarization tensor has been applied in various areas involving electric and electromagnetic areas such as medical imaging and metal detection for object classification purposes [6–10]. Besides that, PT has been used by Ahmad Khairuddin and Lionheart [11] to study the characterization of an object by a weakly electric fish (fish

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with a low discharge of electricity). Examples of the computation of PT based on the explicit formula based on the geometry and the material of the conducting material can be found in [12–15].

Further explanation on the characterization of objects has been discussed in the study conducted by Khairuddin *et al.,* [16], where the researcher highlighted the application of PT to characterize the object. Polarization tensor is usually being implemented because of its properties, which depend on the shape, size, orientation, and material. For instance, Sukri *et al.,* [17] studied the effect of translation of the geometry on the approximated first order PT. The researcher uses the different center of masses of an object, and its results have shown that the approximated PT is independent of their mass of center. An extended method by Bahuriddin *et al.,* [18] in order to determine the semi principle axes of the spheroid which involves prolate and oblate spheroid.

Therefore, in this paper, another property of the first-order PT, which is the effect of an object's size on the approximation of PT, will be studied and will be investigated. Different sizes of different types of geometry will be used to investigate its effect on approximating first-order polarization tensor, sphere, ellipsoid, and cube. Since the analytical solution for first-order polarization tensor is provided only for sphere and ellipsoid, we will first use these results and study the effect of different geometry sizes. We will then compute the numerical result for other geometry, a cube, and from the solution obtained, the effect of different scaling of a cube will be studied. The result obtained is presented, and a discussion on the results is provided.

The paper is organized as follows. First, the mathematical background of the term PT will be reviewed in the next section. From this mathematical background, the method used will be explained. Then, the results obtained that show the effect of the geometry's size on the approximation of the PT will be presented. Lastly, the discussion and conclusion of the results obtained are presented.

2. First-order Polarization Tensor Formulation

Consider a Lipschitz domain where the origin is in the domain $\mathbb{R}^3$, and the conductivity is represented by notation k. The conductivity must satisfy $0 < k \neq 1 < +\infty$. Assume a harmonic function, H in $\mathbb{R}^3$ where $u(x)$ is the solution of the transmission problem

$$\begin{cases} \text{div}(1+(k-1)\chi(B)\text{grad}(u)) = 0 \text{ in } \mathbb{R}^3 \\ u(x) - H(x) = O(|x|^{-2}) \text{ as } |x| \to \infty \end{cases} \tag{1}$$

where χ is the characteristic function of the object in the domain B. Then, from this transmission problem, Ammari and Kang [2] introduced a far-field expansion, which yields to

$$(u-H)(x) = \sum_{|i|,|j|=1}^{+\infty} \frac{(-1)^{|i|}}{i!\,j!} \partial_x^i \Gamma(x) M_{ij}(k,B) \partial^j H(0) \text{ as } |x| \to +\infty \tag{2}$$

where the multi-indices are denoted as i and j while the fundamental solution of the Laplacian is represented as Γ. From Eq. (2), the generalized polarization tensor is denoted as $M_{ij}(k,B)$ and it can also be defined by a system of integral equations over the boundary of B which is

$$M_{ij}(k,B) = \int_{\partial B} y^j \phi_i(y) d\sigma(y) \tag{3}$$

such that $\phi_i(y)$ is defined as

$$\phi_i(y) = \left(\lambda I - \kappa_B^*\right)^{-1}\left(v_x \cdot \nabla x^i\right)(y) \tag{4}$$

where $\lambda = (k+1)/2(k-1)$. Outward normal vector is denoted as v_x while I is the identity. The singular integral operator K_B^* is defined as an integral containing Cauchy Principle Value, $p.v.$

$$\kappa_B^*\phi(x) = \frac{1}{4\pi}\,\text{p.v.}\int_{\partial B} \frac{(x-y)\cdot v_x}{|x-y|^3}\phi(y)d\sigma(y) \tag{5}$$

Throughout this study, we will investigate and compute the first-order polarization tensor based on the equation in Eq. (3), Eq. (4), and Eq. (5).

3. The Analytical Solution for Sphere and Ellipsoid

The analytical solution of first order PT when *B* is an ellipsoid with semi principle axes a, *b*, and *c* that Ammari and Kang [2] have derived is presented by a three by three matrix system, which is

$$M(k,B) = (k-1)|B|\begin{bmatrix} \dfrac{1}{(1-d_1)+kd_1} & 0 & 0 \\[2ex] 0 & \dfrac{1}{(1-d_2)+kd_2} & 0 \\[2ex] 0 & 0 & \dfrac{1}{(1-d_3)+kd_3} \end{bmatrix} \tag{6}$$

where $|B|$ is the volume of the object while d_1, d_2 and d_3 is integrals that can be defined as

$$d_1 = \frac{bc}{a^2}\int_1^{+\infty} \frac{1}{t^2\sqrt{t^2-1+\dfrac{b^2}{a^2}}\sqrt{t^2-1+\dfrac{c^2}{a^2}}}dt,$$

$$d_2 = \frac{bc}{a^2}\int_1^{+\infty} \frac{1}{(t^2-1+\dfrac{b^2}{a^2})^{\frac{3}{2}}\sqrt{t^2-1+\dfrac{c^2}{a^2}}}dt, \tag{7}$$

$$d_3 = \frac{bc}{a^2}\int_1^{+\infty} \frac{1}{\sqrt{t^2-1+\dfrac{b^2}{a^2}}(t^2-1+\dfrac{c^2}{a^2})^{\frac{3}{2}}}dt,$$

where a, b and c are semi principle axes of an ellipsoid. Then, the researchers take the semi principle axes to become similar to each other, yielding to an analytical solution of the sphere where integrals in Eq. (7) are equal to $1/3$. The matrix system in Eq. (6) is then reduced as

$$M(k,B) = (k-1)|B| \begin{bmatrix} \dfrac{3}{2+k} & 0 & 0 \\ 0 & \dfrac{3}{2+k} & 0 \\ 0 & 0 & \dfrac{3}{2+k} \end{bmatrix}. \tag{8}$$

In the next section, we will revise previously theoretical results, which suggest that the first order PT is dependent on the scaling factor of referred objects size. Following that, we will explain the method used for first-order PT approximation, which involves two methods to calculate PT for cube geometry with an unvarying value of conductivity for different sizes of geometry.

4. Methodology

The following proposition is considered from the research conducted by Kang [19], which explains the transformation formula of PT with a scaling factor α.

Proposition 1:

Given that $M(k,B)$ is the first order PT for the referred geometry B. Let α be the scaling factor for the size of B. The first order PT for B after its size is scaled, $M(k,\alpha B)$ satisfies $M(k,\alpha B) = \alpha^3 M(k,B)$.

The above formula explained that we could find the first-order PT for a specified geometry after the size of the geometry is scaled by using the first order PT of the original geometry. This information is crucial in order to validate whether the numerical computation of the first-order PT for an object B with no analytical solution is correct or not. It is more straightforward to verify geometry with provided analytical solutions such as sphere and ellipsoid. For other geometries, we need to verify whether the numerical solution obtained follows the above transformation formula. If the numerical solution obtained followed the transformation formula, we could assume that the solution obtained is correct, and the proposed method can be used to evaluate the first-order PT.

Generally, polarization tensor can be computed by using a numerical method. In this study, the numerical methods used to evaluate the first-order PT are numerical integration with quadratic element and also linear element which involve trapezoidal rule. First and foremost, the geometry is triangularized by a free software, which is Netgen Mesh Generator developed by Joachim Schoberl [20]. For the quadratic element, by setting the meshing to become a second-order element, each triangle will have six vertices, while for the linear element, it will generate three vertices. The information about all triangles will be exported to Matlab software from Netgen for computation purposes.

For the computation of the first-order PT based on linear element integration, we will refer to the procedures by Khairuddin and Lionheart [14], while for quadratic element integration, the computation will follow the steps given by Sukri *et al.,* [21]. The implementation of Gaussian quadrature in the first-order PT computation is needed since we are dealing with an integral equation [22]. The sizes of the mesh (the number of triangles on the surface of the mesh) are changed from coarse to very fine mesh for each geometry. The numerical results of the first order PT for different mesh sizes are evaluated and represented in graphical form to observe its convergence. It is expected that, as the meshes size increased, the numerical solutions of first-order PT will be converge to a

certain value. The next section will review the results obtained by using the aforementioned method together with the scaling factor of each geometry.

5. Results and Discussions

We start discussing the result by presenting the approximation for a sphere with a few scales of sizes. By fixing conductivity to be equal to 1.5 and different radius which are $r = 0.01,\ 0.1, 10$ and 20, the theoretical results for first-order PT is computed. For each sphere (with a different radius), the norm for the first order polarization tensor is presented in Table 1. If

$$M(k,B) = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}, \tag{9}$$

the matrix norm is calculated by using the formula $\|M(k,\beta)\| = \sqrt{M_{11}^2 + \ldots + M_{33}^2}$.

Table 1
The theoretical results for the first order polarization tensor for a sphere when conductivity $k = 1.5$ for a few radius r

r	$\|M(k,\beta)\|$
0.01	0.17952×10^{-5}
0.1	0.17952×10^{-2}
1	0.17952×10^{1}
10	0.17952×10^{4}
20	0.14362×10^{5}
100	0.17952×10^{7}

We set a sphere with a radius 1 to become the reference geometry. The scaling factor, α for other sizes of a sphere, is calculated using the formula r_1 / r_0 where r_1 is the new radius of a sphere, while r_0 is the initial size of a sphere (reference size) which is 1. Therefore, we present the ratio of the norm values of the first order PT with their scaling factor in Table 2. It is observed that the ratio of the norm values of first-order PT, which is based on the reference size of geometry, all are equal to α^3 except when $\alpha = 20$ but the ratio is still almost equal to α^3. This is due to the rounding in decimal places for the value $3/(2+k)$ in the first order PT given by Eq. (8).

Table 2
The ratio of the norm of the theoretical results for the first order polarization tensor for an ellipsoid with the scaling factor, α

α	$\dfrac{\|M(k,\alpha\beta)\|}{\|M(k,\beta)\|}$
0.01	10^{-6}
0.1	10^{-3}
10	10^{3}
20	8000.22
100	10^{6}

Similar conductivity has been used in the computation of the first order polarization tensor for an ellipsoid where the conductivity is 1.5. For ellipsoid, we used different lengths of semi principle axes of a, b and c denoted by (a,b,c) which are, $(0.01, 0.02, 0.03)$, $(0.1, 0.2, 0.3)$, $(1, 2, 3)$ and $(2, 4, 6)$ where the reference axis is set to be $(1, 2, 3)$. The norm computed using the theoretical results for the first-order PT for an ellipsoid are presented in Table 3.

Table 3
The norm for the theoretical results for the first order polarization tensor for an ellipsoid with fixed conductivity $k = 1.5$ and different sizes of an object

| Semi principle axis, (a,b,c)	$\|M(k,\beta)\|$
$(0.01, 0.02, 0.03)$	1.1655×10^{-5}
$(0.1, 0.2, 0.3)$	0.0117
$(1, 2, 3)$	11.6554
$(2, 4, 6)$	93.2439

Table 4 depicted the ratio of the norm for the first-order PT with respect to the first order PT of the reference ellipsoid. It is observed that each ratio for the norm of the first-order PT, which is based on the reference ellipsoid $(1, 2, 3)$, is approximately equal to α^{3}. Two ratios are not exactly equal to α^{3} due to Eq. (7) has been evaluated by numerical integration, which will cause a small error in the first order PT for every ellipsoid.

Table 4
The ratio of the norm of the theoretical results for the first order polarization tensor for an ellipsoid with the scaling factor, α

α	$\dfrac{\|M(k,\alpha\beta)\|}{\|M(k,\beta)\|}$
0.01	9.9997×10^{-7}
0.1	1.0040×10^{-3}
2	8.0000

Next, we will consider the results for cube and observe whether the result obtained using the proposed methods satisfy Proposition 1. Figure 1, Figure 2, and Figure 3 show the norm values of the first-order PT for cubes at conductivitiy also equal to 1.5. Here, the dimension of the cube is set to be $0.2 \times 0.2 \times 0.2$ (see Figure 1), $0.4 \times 0.4 \times 0.4$ (see Figure 2), and $0.6 \times 0.6 \times 0.6$ (see Figure 3). From Figure 1, we can observe that, as the total number of surface elements increased, the norm of the first-order PT will eventually lead to a specified norm of the first-order PT. However, the numerical results for the first-order PT show a stable solution when using quadratic elements. From the graph in Figure 1(b), from $N = 424$ to $N = 1696$, the norm values of the first-order PT have less difference than the norm values shown by the first-order PT using linear element in Figure 1(a).
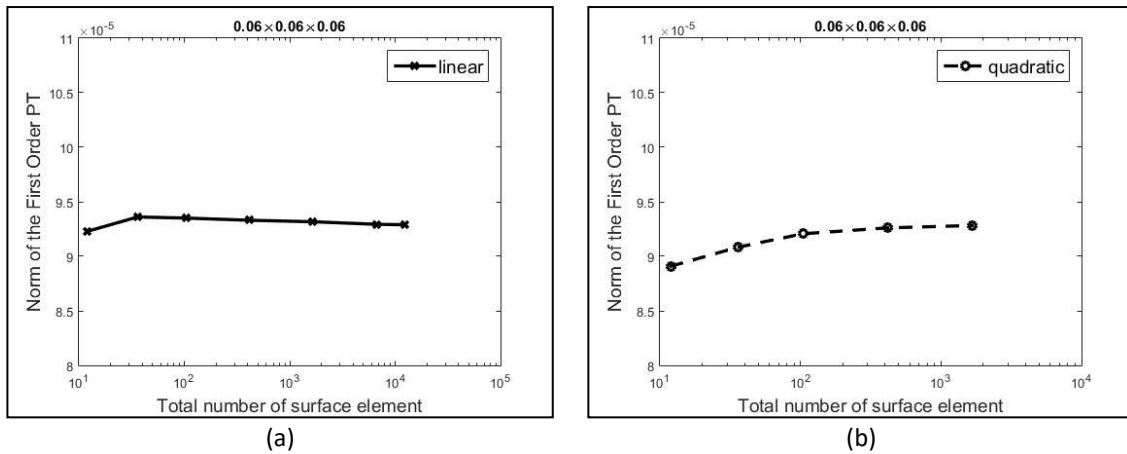

| (a) | (b) |
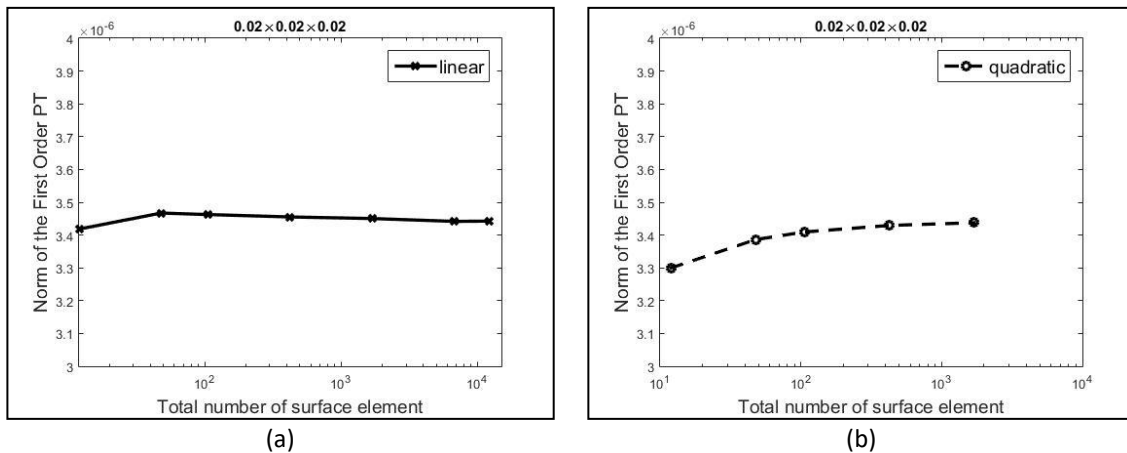
Fig. 1 Norm of the first-order PT for cube with the dimension $0.2 \times 0.2 \times 0.2$ with conductivity, $k = 1.5$ obtained by using (a) linear element with the total number of the surface elements are $N = 12, 48, 106, 424, 6784$ and 12288 (b) quadratic element integration with the total number of the surface elements are $N = 12, 48, 106, 424,$ and 1696

Next, for the norm for the first-order PT with dimension $0.4 \times 0.4 \times 0.4$ and $0.6 \times 0.6 \times 0.6$, based on Figure 2 and Figure 3, the graphs' pattern followed graph's pattern for a cube with dimension $0.2 \times 0.2 \times 0.2$.

| (a) | (b) |
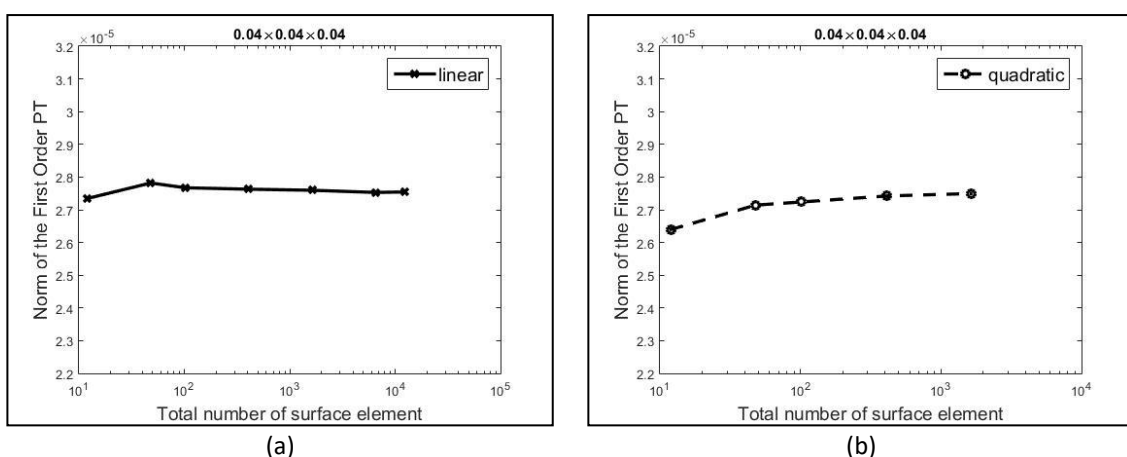
Fig. 2 Norm of the first-order PT for cube with the dimension $0.4 \times 0.4 \times 0.4$ with conductivity, $k = 1.5$ obtained when using (a) linear element with the total number of the surface elements are $N = 12, 48, 102, 408, 6528$ and 12288 (b) quadratic element integration with the total number of the surface elements are $N = 12, 48, 102, 408,$ and 1632

Fig. 3 Norm of the first-order PT for cube with the dimension $0.6 \times 0.6 \times 0.6$ with conductivity, $k = 1.5$ obtained using (a) linear element with the total number of the surface elements are $N = 12, 36, 104, 416, 6656$ and 12288 (b) quadratic element integration with the total number of the surface elements are $N = 12, 36, 104, 416$, and 1664

Next, for scaling purposes, we need to set one size of cube to be our reference geometry. Here, our reference geometry is cube with dimensions $0.4 \times 0.4 \times 0.4$. Hence scaling factor α for cube with dimension $0.2 \times 0.2 \times 0.2$ is equal to $1/2$ while for cube with dimension, $0.6 \times 0.6 \times 0.6$, the scale is $3/2$. Table 5 showed the numerical results of the norm of the first-order PT for the cube with different dimensions. In this table, the total number of surface elements used for all cubes are inconsistent since they are automatically generated by Netgen.

Table 5
The norm of the first-order polarization tensor for the cube with dimension $0.2 \times 0.2 \times 0.2$, $0.4 \times 0.4 \times 0.4$ and $0.6 \times 0.6 \times 0.6$ by using the linear and quadratic element integration

Dimension	Total number of the surface element, N	Norm of first-order PT (linear), $\left\| M_L(k,\beta) \right\|$	Norm of first-order PT (quadratic), $\left\| M_Q(k,\beta) \right\|$
$0.2 \times 0.2 \times 0.2$	1696	3.4503×10^{-6}	3.4376×10^{-6}
$0.4 \times 0.4 \times 0.4$	1632	2.7597×10^{-5}	2.7496×10^{-5}
$0.6 \times 0.6 \times 0.6$	1664	9.3170×10^{-5}	9.2825×10^{-5}

Table 6 then showed the comparison between the ratio for the norm of the first-order PT obtained from quadratic and linear element integrations for the cubes together with their scaling factor. From Proposition 1, the ratio of the norm values of first-order PT must be equal to α^3. It can be observed that, for a cube with dimension $0.6 \times 0.6 \times 0.6$, the ratio 3.3759 obtained when quadratic element integration is used is closer to the value $\alpha^3 = 3.375$ if compared to the linear element, which is 3.3761. Nevertheless, both methods showed a similar ratio, for $0.2 \times 0.2 \times 0.2$ which are 0.125.

Table 6
The comparison between the ratio of norm of the first-order polarization tensor for the cubes with dimension $0.2\times0.2\times0.2$ and $0.6\times0.6\times0.6$ based on the referred cube with dimension $0.4\times0.4\times0.4$ the computed first order where $k=1.5$ for linear and quadratic element

Dimension	The scaling factor, α	α^3	$\dfrac{\|M_L(k,\alpha\beta)\|}{\|M_L(k,\beta)\|}$	$\dfrac{\|M_Q(k,\alpha\beta)\|}{\|M_Q(k,\beta)\|}$
$0.2\times0.2\times0.2$	$\dfrac{1}{2}$	0.1250	0.1250	0.1250
$0.6\times0.6\times0.6$	$\dfrac{3}{2}$	3.375	3.3761	3.3759

6. Conclusions

In this study, we have shown that the first order PT computed using both theoretical and numerical methods follow the transformation property as stated in Proposition 1 where, the first order PT of an object is dependent on the size of the object. Specifically, the scaling factor used on the object can be calculated based on the first order PT of the referred object and the first order PT of the scaled object. In addition, for the first order PT for cube, approximated by using numerical method, we have included the results for both linear and quadratic element integration using graphical and tabulation representation. It can be concluded that, as the number of surface element increased, the numerical solution of the first order PT converged to a certain value. As our results show that the first order PT for all geometries considered satisfy Proposition 1, it is also suggested to compute the first order PT for another geometry by the proposed numerical method before observing whether the obtained solutions also follow Proposition 1 in order to further investigate as well as improving the performance of the numerical method used.

Acknowledgment

References
[1] Pólya, G. "A minimum problem about the motion of a solid through a fluid." *Proceedings of the National Academy of Sciences of the United States of America* 33, no. 7 (1947): 218-221. https://doi.org/10.1073/pnas.33.7.218
[2] Ammari, Habib, and Hyeonbae Kang. *Polarization and moment tensors: with applications to inverse problems and effective medium theory*. Vol. 162. Springer Science & Business Media, 2007.
[3] Ammari, Habib, Hyeonbae Kang, Hyundae Lee, and Mikyoung Lim. "Enhancement of near cloaking using generalized polarization tensors vanishing structures. Part I: The conductivity problem." *Communications in Mathematical Physics* 317, no. 1 (2013): 253-266. https://doi.org/10.1007/s00220-012-1615-8
[4] Ammari, Habib, Michael S. Vogelius, and Darko Volkov. "Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities of small diameter II. The full Maxwell equations." *Journal de Mathématiques Pures et Appliquées* 80, no. 8 (2001): 769-814. https://doi.org/10.1016/S0021-7824(01)01217-X
[5] Ammari, Habib, Junqing Chen, Zhiming Chen, Darko Volkov, and Han Wang. "Detection and classification from electromagnetic induction data." *Journal of Computational Physics* 301 (2015): 201-217. https://doi.org/10.1016/j.jcp.2015.08.027
[6] Dekdouk, Bachir, Liam A. Marsh, David W. Armitage, and Anthony J. Peyton. "Estimating magnetic polarizability tensor of buried metallic targets for land mine clearance." In *Ultra-Wideband, Short-Pulse Electromagnetics* 10, pp. 425-432. Springer, New York, NY, 2014. https://doi.org/10.1007/978-1-4614-9500-0_38

[7]   Ledger, Paul D., and William RB Lionheart. "Characterizing the shape and material properties of hidden targets from magnetic induction data." *The IMA Journal of Applied Mathematics* 80, no. 6 (2015): 1776-1798. https://doi.org/10.1093/imamat/hxv015

[8]   Ammari, Habib, Junqing Chen, Zhiming Chen, Josselin Garnier, and Darko Volkov. "Target detection and characterization from electromagnetic induction data." *Journal de Mathématiques Pures et Appliquées* 101, no. 1 (2014): 54-75. https://doi.org/10.1016/j.matpur.2013.05.002

[9]   Ammari, Habib, Junqing Chen, Zhiming Chen, Darko Volkov, and Han Wang. "Detection and classification from electromagnetic induction data." *Journal of Computational Physics* 301 (2015): 201-217. https://doi.org/10.1016/j.jcp.2015.08.027

[10]   Yunos, Nurhazirah Mohamad, Taufiq Khairi Ahmad Khairuddin, and William RB Lionheart. "Identification of a Spheroid based on the First Order Polarization Tensor." *Journal of Science and Technology* 9, no. 3 (2017): 154-159.

[11]   Khairuddin, Taufiq K. Ahmad, and William RB Lionheart. "Characterization of objects by electrosensing fish based on the first order polarization tensor." *Bioinspiration & Biomimetics* 11, no. 5 (2016): 055004. https://doi.org/10.1088/1748-3190/11/5/055004

[12]   Khairuddin, Taufiq Khairi Ahmad, and William RB Lionheart. "Computing the first order polarization tensor: welcome BEM++!." *Menemui Matematik (Discovering Mathematics)* 35, no. 2 (2013): 15-20.

[13]   Ledger, Paul D., and WR Bill Lionheart. "Understanding the magnetic polarizability tensor." *IEEE Transactions on Magnetics* 52, no. 5 (2015): 1-16. https://doi.org/10.1109/TMAG.2015.2507169

[14]   Khairuddin, Taufiq Khairi Ahmad, and William Lionheart. "Some properties of the first order polarization tensor for 3-D domains." *MATEMATIKA: Malaysian Journal of Industrial and Applied Mathematics* 29, no. 1 (2013): 1-18.

[15]   Khairuddin, Taufiq K. A., and William R. B. Lionheart. "Numerical comparisons for the approximated first order polarization tensor for ellipsoids." *Applied Mathematics and Computational Intelligence* 4, no. 1 (2015): 341-354.

[16]   Khairuddin, Taufiq Khairi Ahmad, Nurhazirah Mohamad Yunos, and Suzarina Ahmed Sukri. "Characterization of Objects based on the Polarization Tensor: Nature versus Artificial Intelligence." In *IOP Conference Series: Materials Science and Engineering*, vol. 1051, no. 1, p. 012033. IOP Publishing, 2021. https://doi.org/10.1088/1757-899X/1051/1/012033

[17]   Sukri, Suzarina Ahmed, Taufiq Khairi Ahmad Khairuddin, and Yeak Su Hoe. "The Effect of Translation on the Approximated First Order Polarization Tensor of Sphere and Cube." *Open Journal of Science and Technology* 3, no. 3 (2020): 274-282.

[18]   Bahuriddin, Nur Safirah, Mukhiddin Muminov, Taufiq Khairi Ahmad Khairuddin, Syafina Ahmad, and Wan Rohaizad Wan Ibrahim. "An Extended Method for Fitting the First Order Polarization Tensor to a Spheroid." *Journal of Advanced Research in Applied Sciences and Engineering Technology* 23, no. 1 (2021): 8-17. https://doi.org/10.37934/araset.23.1.817

[19]   Kang, Hyeonbae. "Layer potential approaches to interface problems." *Inverse problems and imaging, Panoramas et Syntheses, Societe Mathematique de France, to appear* (2015).

[20]   Schöberl, Joachim. "NETGEN An advancing front 2D/3D-mesh generator based on abstract rules." *Computing and Visualization in Science* 1, no. 1 (1997): 41-52. https://doi.org/10.1007/s007910050004

[21]   Sukri, Suzarina Ahmed, Yeak Su Hoe, and Taufiq Khairi Ahmad Khairuddin. "Quadratic Element Integration of Approximated First Order Polarization Tensor for Sphere." *Malaysian Journal of Fundamental and Applied Sciences* 16, no. 5 (2020): 560-565.

[22]   Huang, Cheng-Hung. "Hypersingular boundary integral equation for axisymmetric elasticity." *International Journal for Numerical Methods in Engineering* 52, no. 11 (2001): 1337-1354. https://doi.org/10.1002/nme.259