



Numerical Solution for Boundary Layer Flow of a Dusty Micropolar Fluid Due to a Stretching Sheet with Constant Wall Temperature

Anisah Dasman^{1,2,*}, Abdul Rahman Mohd Kasim^{1,*}, Iskandar Waini³, Najiyah Safwa Khashi'ie³

¹ Pusat Sains Matematik, Universiti Malaysia Pahang, Lebuhraya Tun Razak, 26300 Gambang, Kuantan, Pahang, Malaysia

² Fakulti Sains Komputer and Matematik, Universiti Teknologi MARA Cawangan Pahang, Kampus Jengka, 26400 Bandar Pusat Jengka, Pahang, Malaysia

³ Fakulti Teknologi Mekanikal dan Pembuatan, Universiti Teknikal Malaysia Melaka, Malaysia

ARTICLE INFO

Article history:

Received 10 May 2021

Received in revised form 15 August 2021

Accepted 20 August 2021

Available online 5 September 2021

Keywords:

Dusty micropolar fluid; two phase fluids;
bvp4c function; stretching sheet;
numerical solution

ABSTRACT

This paper aims to present the numerical study of a dusty micropolar fluid due to a stretching sheet with constant wall temperature. Using the suitable similarity transformation, the governing partial differential equations for two-phase flows of the fluid and the dust particles are reduced to the form of ordinary differential equations. The ordinary differential equations are then numerically analysed using the *bvp4c* function in the MATLAB software. The validity of present numerical results was checked by comparing them with the previous study. The results graphically show the numerical solutions of velocity, temperature, and microrotation distributions for several values of the material parameter K , fluid-particle interaction parameter β and Prandtl number for both fluid and dust phase. The effect of microrotation is investigated and analysed as well. It is found that the distributions are significantly influenced by the investigated parameters for both phases.

1. Introduction

In this realistic life, the fluid state is not completely pure. The fluid will contain small dust particles. Consequently, the study of two-phase flow is becoming increasingly important, especially in engineering and industry. In addition to its importance in petroleum engineering and crude oil industry such as crude oil refining and sedimentation process, two-phase flow also occurs naturally, such as haze flow, mudflow in rivers, blood flow in body systems, and more. Two-phase flow is also the simplest flow model in multi-phase flow. The two-phase flow model was introduced by Saffman [1] and received a lot of attention because of its importance. Moreover, Aggarwal and Gupta [2] studied the micropolar fluid with dust/suspended particles on an electrically conducted heated layer, dissolved from below in the presence of a uniform vertical magnetic field in a porous medium. The problem of magnetohydrodynamic flow and heat transfer of a viscous, incompressible, and

* Corresponding author.

E-mail address: anisahd@uitm.edu.my

* Corresponding author.

E-mail address: rahmanmohd@ump.edu.my

electrically conducting dusty fluid over an unsteady stretching sheet is analysed numerically by Manjunatha and Gireesha [3]. Besides that, Siddiqua *et al.*, [4] conducted a detailed investigation of a dusty fluid's two-dimensional natural convection flow. Izani and Ali [5] followed this, which deals with an incompressible viscous dusty fluid over an exponentially stretching surface with an exponential temperature distribution in convective heat transfer characteristics. Then, Siddiqua *et al.*, [6] used a modified power-law viscosity model to present a boundary-layer analysis of two-phase dusty non-Newtonian fluid flow along a vertical surface. The effect of an aligned magnetic field is numerically analysed for mixed convection flow of dusty Casson fluid over a stretching sheet studied by Arifin *et al.*, [7]. It shows that the understanding of a two-phase flow behaviour can help predict the flow processes of the polluted fluid. The unsteady characteristics of the mixed convection boundary layer flow of dusty fluid past a vertical wedge were conducted by Hossain *et al.*, [8]. It was solved by two distinct methods. For the entire frequency range, the straightforward finite difference method is used, while for the low-frequency range, the asymptotic series expansion method was employed. Significant impact in the fluid temperature distribution with an energy source of heat generation applied to dusty Casson fluid was studied by Kasim *et al.*, [9]. Moreover, Arifin *et al.*, [10] revealed that the fluid particle interaction parameter influencing the fluid velocity decreased the fluid motion in mixed convection flow of dusty Williamson fluid with aligned magnetic field over a vertical stretching sheet. Apart from that, the latest investigation on the dusty fluid can be found in [11].

Motivated to the significance and imperative application towards boundary layer flow of a dusty micropolar fluid, this study investigates the dusty micropolar fluid due to stretching sheet using the theory of boundary layer and Constant Wall Temperature (CWT). Numerical analysis was conducted using the `bvp4c` function build in MATLAB software.

2. Problem Formulation

Consider the steady, incompressible two-dimensional boundary layer flow of dusty micropolar fluid due to stretching sheet with constant wall temperature. The factors considered are dust particles assumed in spheres where the size and density are uniform throughout the flow. Under the boundary layer and Boussinesq approximations, the governing equations contained mass, momentum and energy equations for two-phase flow given by Qasim *et al.*, [18] and Turkyilmazoglu [19], which can be written as

Fluid phase,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \left(\nu + \frac{\kappa}{\rho} \right) \frac{\partial^2 u}{\partial y^2} + \frac{\kappa}{\rho} \frac{\partial N}{\partial y} + \frac{\rho_p}{\rho \tau_m} (u_p - u) \quad (2)$$

$$\rho j \left(u \frac{\partial N}{\partial x} + v \frac{\partial N}{\partial y} \right) = \gamma \frac{\partial^2 N}{\partial y^2} - \kappa \left(2N + \frac{\partial u}{\partial y} \right) \quad (3)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \frac{\rho_p c_s}{\tau_T \rho c_p} (T_p - T) \quad (4)$$

Dust phase,

$$\frac{\partial u_p}{\partial x} + \frac{\partial v_p}{\partial y} = 0 \quad (5)$$

$$u_p \frac{\partial u_p}{\partial x} + v_p \frac{\partial u_p}{\partial y} = -\frac{1}{\tau_m} (u_p - u) \quad (6)$$

$$u_p \frac{\partial T_p}{\partial x} + v_p \frac{\partial T_p}{\partial y} = -\frac{1}{\tau_T} (T_p - T) \quad (7)$$

where (u, v) and (u_p, v_p) are velocities components of the fluid and dust phase along x and y axes, respectively. ν is the kinematic viscosity, κ is the vortex viscosity, ρ and ρ_p are the density of fluid and dust phase, respectively, N is the component of micro-rotation vector normal to the x and y axes, τ_m is called velocity relaxation time of particle phase and τ_T is the thermal relaxation time of particle phase. Next, j is a microinertia density, γ is spin gradient viscosity, T and T_p are the temperature of fluid and dust phase, respectively, α is the thermal diffusivity, while c_p and c_s are specific heat of fluid and dust phase, respectively. The governing equation above is solved together with the following boundary conditions

$$u = u_w(x) = ax, \quad v = V_w, \quad N = -n \frac{\partial u}{\partial y}, \quad T = T_w \quad \text{at } y = 0, \quad (8)$$

$$u \rightarrow 0, \quad N \rightarrow 0, \quad u_p \rightarrow 0, \quad v_p \rightarrow v, \quad T \rightarrow T_\infty, \quad T_p \rightarrow T_\infty \quad \text{as } y \rightarrow \infty.$$

The following similarity transformations are introduced to reduce the governing Eq. (1) to (7) accompanied by boundary conditions (8), given by

$$u = axf'(\eta), \quad v = -(av)^{\frac{1}{2}} f(\eta), \quad \eta = \left(\frac{a}{v}\right)^{\frac{1}{2}} y, \quad \psi = (va)^{\frac{1}{2}} xf(\eta), \quad N = ax \left(\frac{a}{v}\right)^{\frac{1}{2}} h(\eta), \quad (9)$$

$$\theta = \frac{T - T_\infty}{T_w - T_\infty}, \quad u_p = axF'(\eta), \quad v_p = -(av)^{\frac{1}{2}} F(\eta), \quad \theta_p = \frac{T_p - T_\infty}{T_w - T_\infty}$$

Here, ψ is the stream function explicated as $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$, which identically satisfy Eq. (1).

Then, the transformed ordinary differential equations can be expressed as

$$(1 + K)f''' + ff'' - f'^2 + Kh' + \beta L(F' - f') = 0 \quad (10)$$

$$(2 + K)h'' + fh' - fh - K(2h + f'') = 0 \quad (11)$$

$$\frac{1}{\text{Pr}} \theta'' + f \theta' + \frac{2}{3} \beta L (\theta_p - \theta) = 0 \quad (12)$$

$$F'^2 - FF'' + \beta(F' - f') = 0 \quad (13)$$

$$\theta_p' F + \frac{2}{3} \frac{\beta}{\gamma \text{Pr}} (\theta - \theta_p) = 0 \quad (14)$$

Meanwhile, the boundary conditions (8) are reduced to,

$$\begin{aligned} f(0) = s, f'(0) = 1, h(0) = -nf''(0), \theta(0) = 1(\text{CWT}) \text{ at } \eta=0 \\ f'(\eta) \rightarrow 0, h(\eta) \rightarrow 0, F(\eta) \rightarrow f(\eta), F'(\eta) \rightarrow 0, \theta(\eta) \rightarrow 0, \theta_p(\eta) \rightarrow 0 \text{ as } \eta \rightarrow \infty \end{aligned} \quad (15)$$

where dynamic viscosity $\mu = \nu\rho$, material parameter $K = \frac{\kappa}{\mu} = \frac{\kappa}{\nu\rho}$, $L = \frac{\rho_p}{\rho}$ is the mass concentration of the particle phase and $\beta = \frac{1}{a\tau_m}$ is the fluid-particle interaction parameter. Moreover, $\text{Pr} = \frac{\nu}{a}$ is a

Prandtl number, $\tau_T = \frac{3}{2} \gamma \tau_m \text{Pr}$ is the relation between thermal relaxation time and velocity relaxation

time, $\gamma_T = \frac{c_s}{c_p}$ is the specific heat ratio of mixture, $h(\eta)$ is the similarity micro-rotation or angular

velocity, s is the wall permeability parameter with wall suction $s < 0$ and wall blowing $s > 0$, while n represents a dimension for the concentration of microelements. Also, $n=0$ denotes the weak microelement concentrations without the ability of rotation of microelements. On the other hand, weak microelement concentrations with stress tensor having no antisymmetric component are denoted by $n = \frac{1}{2}$. Besides, $n=1$ means the boundary layer flow is in a turbulent state.

3. Numerical Procedure

Steady, two-dimensional boundary layer flow of dusty micropolar fluid due to stretching sheet is considered. The system of nonlinear Eq. (1) to (7) along with boundary conditions (8) are constructed into a form of nonlinear ordinary differential equations using similarity transformation. Then, to seek the numerical solution of the problem, the nonlinear differential Eq. (10) to (14) and boundary conditions (15) are interpreted numerically using bvp4c functions build in MATLAB software. Table 1 shows the comparison of $-\left(1 + \frac{K}{2}\right) f''(0)$ for various values of the material parameter, K when

$n = \frac{1}{2}$ and $\gamma \rightarrow \infty$, ignoring the presence of dust effect, where $s = \beta = L = 0$. However, Table 2 presents the comparison of $-\theta'(0)$ for various values of Pr at $K = s = \beta = L = 0$ with constant wall temperature.

Table 1

Comparison of $-\left(1 + \frac{K}{2}\right) f''(0)$ for various values of K at $n = \frac{1}{2}$ and $s = \beta = L = 0$ at $\gamma \rightarrow \infty$

K	Qasim <i>et al.</i> , [18]	Turkyilmazoglu [19]	Present
0	-1.000000	-1.00000000	-1.000000000
1	-1.224741	-1.22474487	-1.224744871
2	-1.414218	-1.41421356	-1.414213562
4	-1.732052	-1.73205081	-1.732050811

Table 2

Comparison of $-\theta'(0)$ for various values of Pr at $K = s = \beta = L = 0$ with constant wall temperature

Pr	Qasim <i>et al.</i> , [18]	Turkyilmazoglu [19]	Present
0.72	0.46360	0.46314456	0.463144560
1	0.58202	0.58197671	0.581976709
3	1.16525	1.16524595	1.165245953
5	1.56805	1.56805419	1.568054194
7	1.89542	1.89540326	1.895403263
10	2.30800	2.30800394	2.308003946
100	7.75826	7.76565169	7.765651692

This clearly specifies that the past numerical results [18-19] correspond to the available numerical solutions for the neglected dusty effect case. It is observed that the values increase with increasing values of K . Meanwhile, an increase in the values of Pr causes the decreasing values of $\theta'(0)$.

4. Result and Discussion

The dusty micropolar fluid has two parts on its governing equations, which are fluid and dust phases. The velocity $(f'(\eta), F'(\eta))$ and the temperature $(\theta(\eta), \theta_p(\eta))$ profile for various values of physical parameters are depicted graphically. Figure 1 and Figure 2 show the velocity profile against η for various values of K with $n = 1$ and $n = \frac{1}{2}$, respectively for both fluid and dust phase. It is noticed from both figures that the velocity profile for fluid phase and dust phase decreases as the values of K increases. This is because the fluid begins to behave like a rigid body for a bigger value of K and velocity profile difference pattern according to microelement concentrations, n .

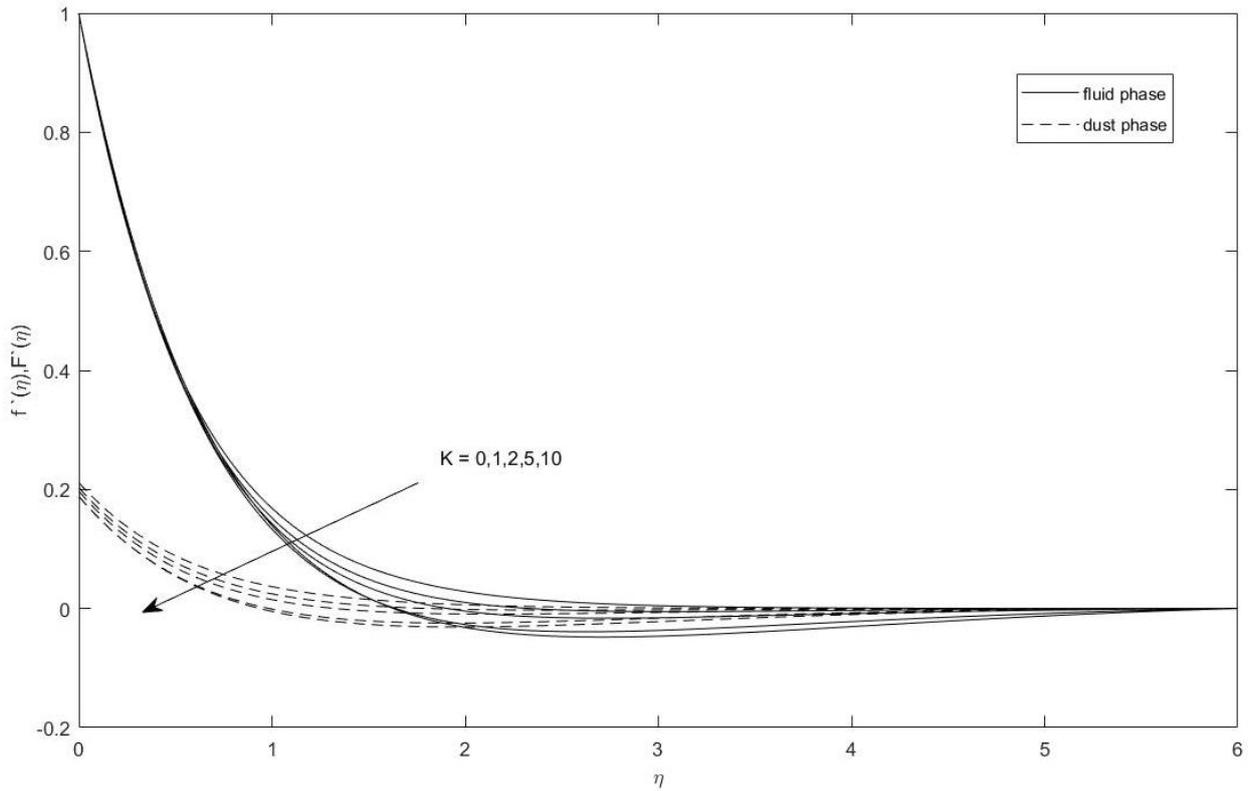


Fig. 1. Velocity profile against η for various values of K with $n = 1, \lambda = 1, s = 1, \beta = 0.5, L = 1$ and $Pr = 7$

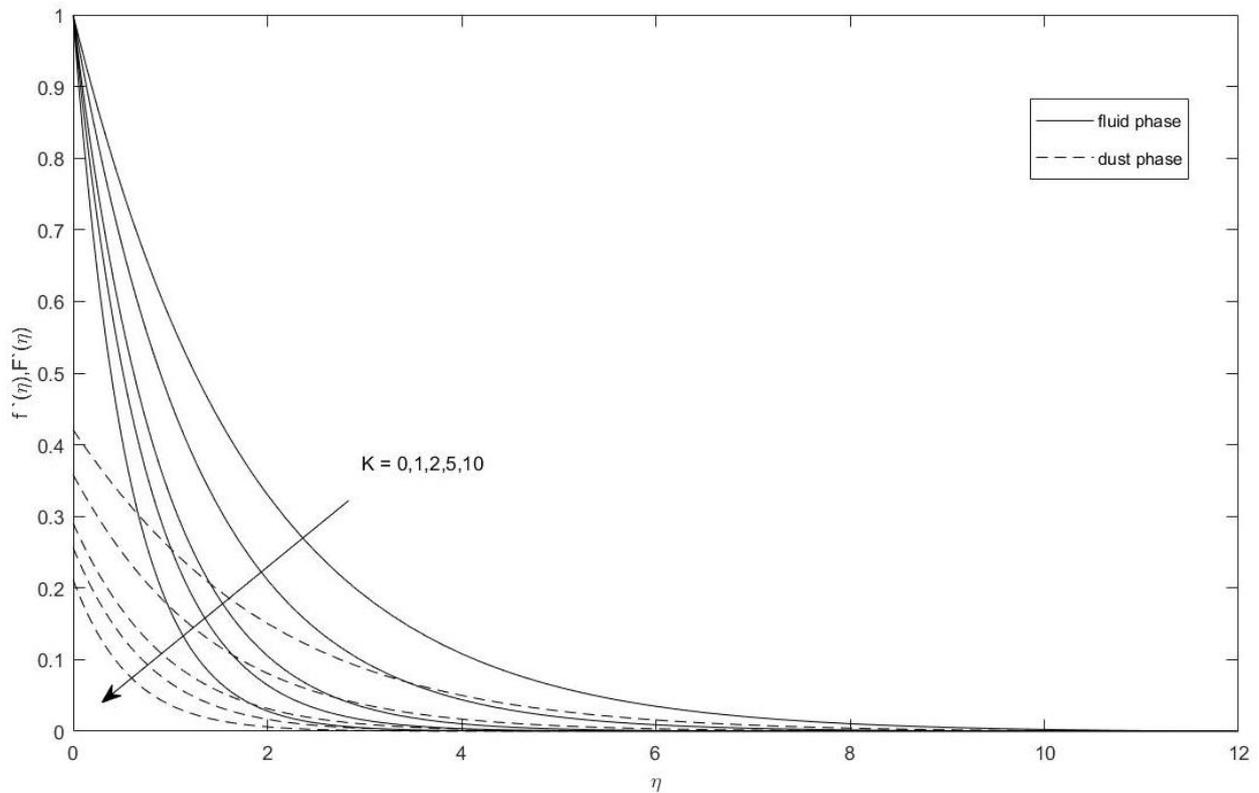


Fig. 2. Velocity profile against η for various values of K with $n = \frac{1}{2}, \lambda = 1, s = 1, \beta = 0.5, L = 1, \gamma = 1$ and $Pr = 7$

Figure 3 and Figure 4 depict the variation of velocity profiles against η for different values of the local fluid-particle interaction parameter β for both fluid and dust phase velocity. From this figure, we infer that the increasing values of β decreases the velocity profile fluid phase and increases the dust particle phase in the boundary layer. Meanwhile, there are no differences in velocity profile pattern according to microelement concentrations, n .

The temperature profile for various values of fluid-particle interaction β is presented for both phases in Figure 5. This figure demonstrates that the escalating values of β lessens the temperature profile fluid phase and increases the dust particle phase. Figure 6 explains the temperature profile for different values of Prandtl number, Pr. From this figure, an increase in the value of Pr decreases the temperature fluid and dust phase. In other words, the rate of cooling can be increased using the Prandtl number.

Figure 7 represents the microrotation distribution for various values of K . The increased values of K shows that the microrotation continuously decreases with η and becomes zero. As expected, the microrotation effect is most influential near the wall. Table 3 provides the values of angular velocity $-h(\eta)$ for different values of s . It can be said that the higher the value of s , the lower the value of the angular velocity.

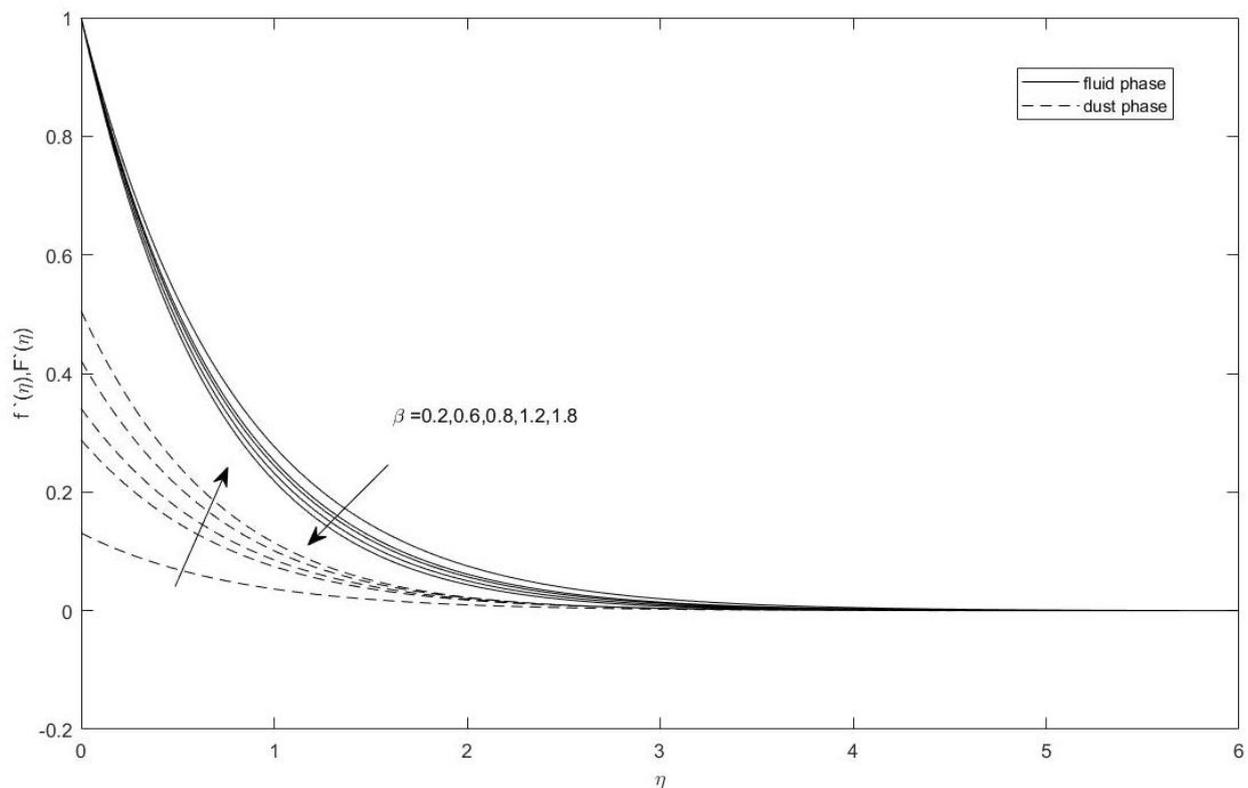


Fig. 3. Velocity profile against η for various values of β with $n = 1, \lambda = 1, s = 1, \gamma = 1, L = 1, K = 1$ and $Pr = 7$

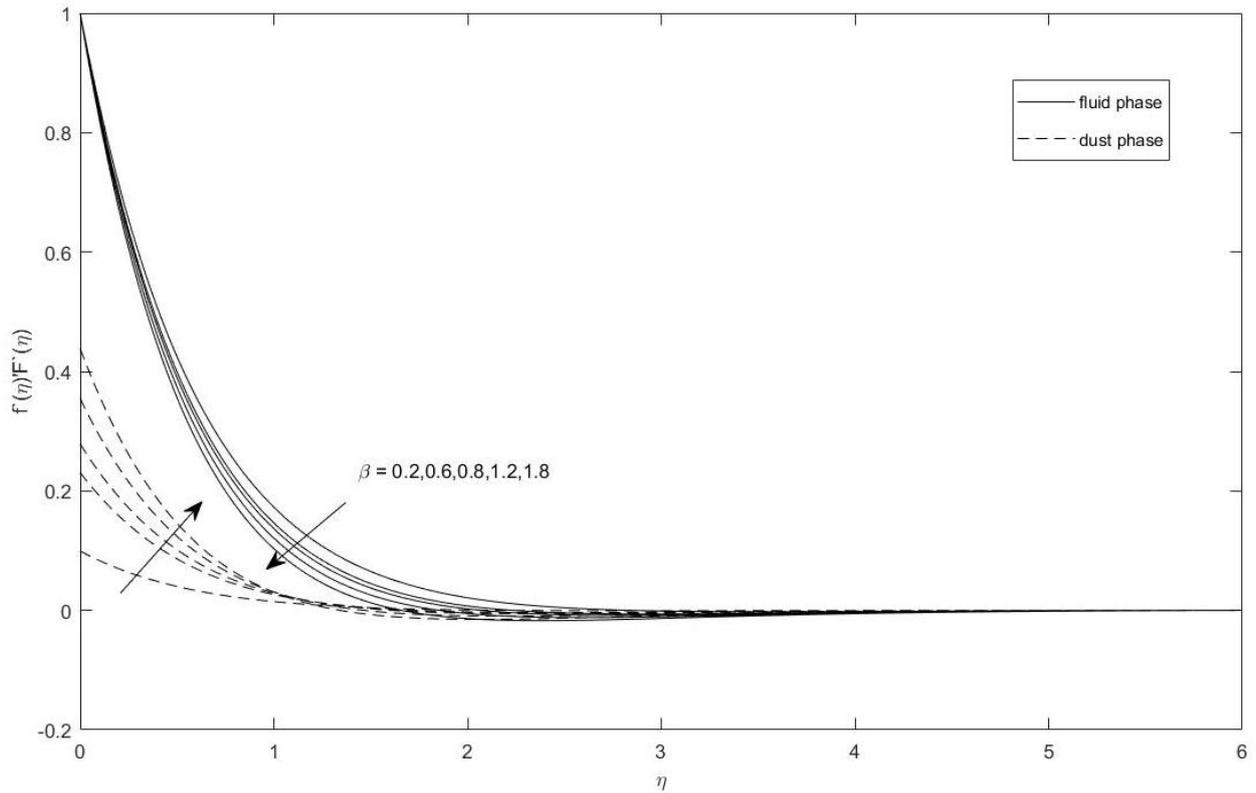


Fig. 4. Velocity profile against η for various values of β with $n = \frac{1}{2}, \lambda = 1, s = 1, K = 1, L = 1, \gamma = 1$ and $Pr = 7$

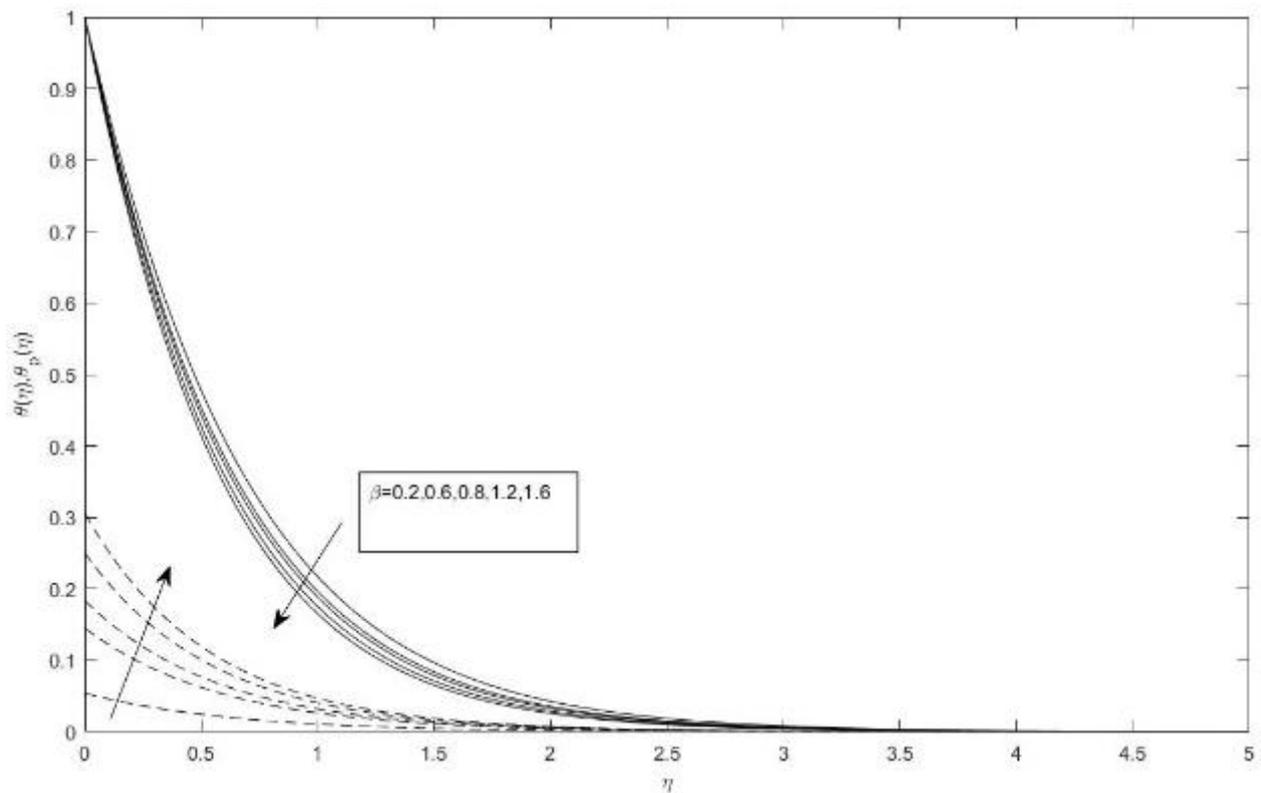


Fig. 5. Temperature profile against η for various values of β with $n = 1, \lambda = 1, s = 1, \gamma = 1, L = 1, K = 1$ and $Pr = 1$

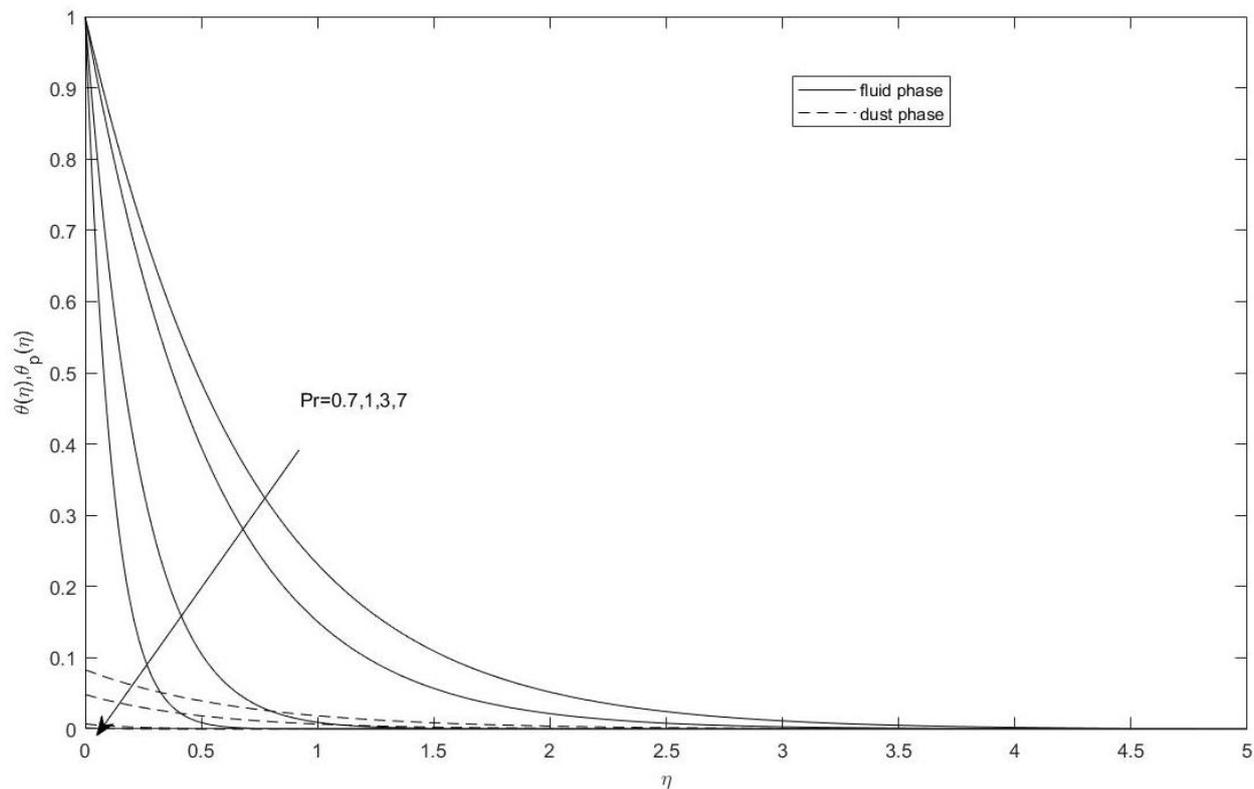


Fig. 6. Temperature profile against η for various values of Pr with $n = 1, \lambda = 1, s = 1, \gamma = 1, L = 1, K = 1$ and $\beta = 0.2$

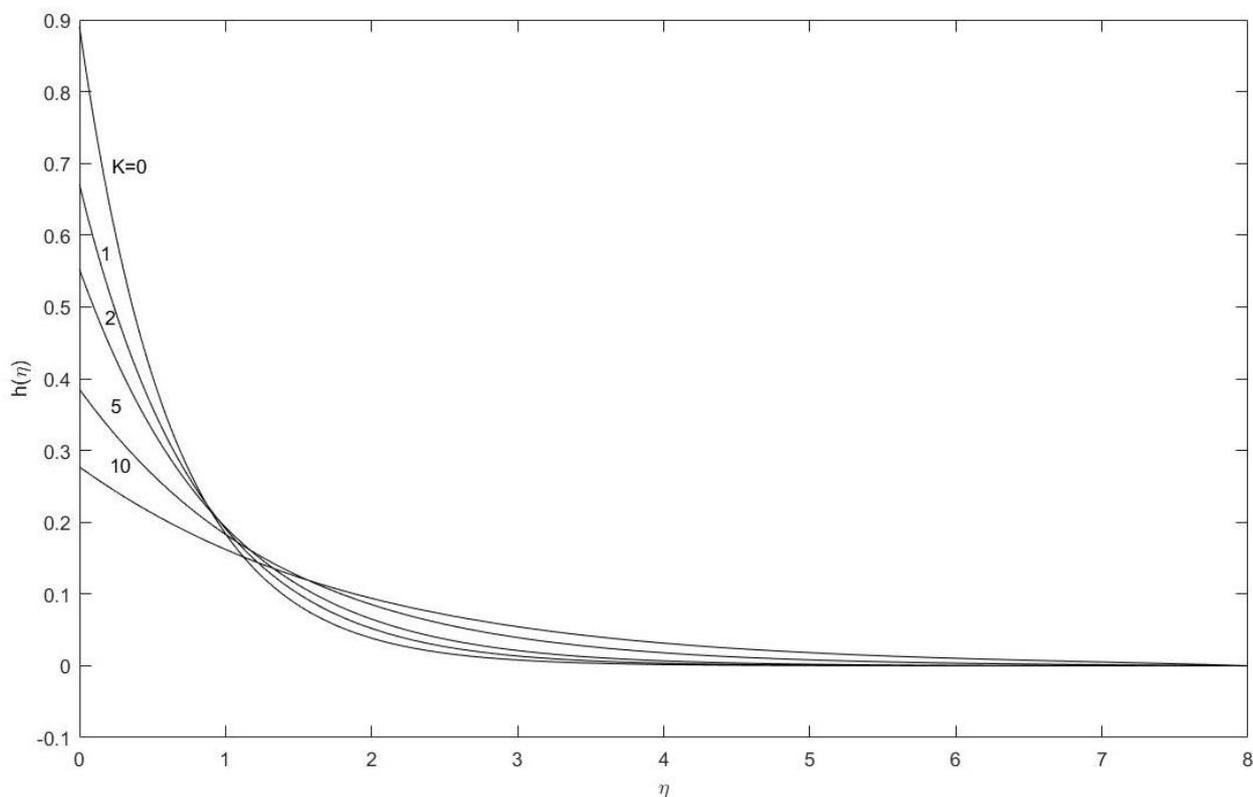


Fig. 7. Microrotation distribution for various values of K with $n = \frac{1}{2}, \lambda = 1, s = 1, \beta = 0.5, L = 1, \gamma = 1$ and $Pr = 7$

Table 3
Computation of $-h(\eta)$ for various values of s

Suction, s	$-h(\eta)$
-4	0.117375254
-1	0.198520464
0	0.558289728
1	1.060974445
2	1.651629633
3	2.288236165
4	2.948671671

5. Conclusion

The present study investigates the effect of dust particle suspension in a micropolar fluid. The governing partial differential equations for the two-phase flow of the fluid and the dust particles are driven to the form of ordinary differential equations using appropriate similarity transformation. The ordinary differential equations were then numerically analysed using the `bvp4c` function in the MATLAB software. The velocity profile for fluid phase and dust phase decreases with increasing micropolar fluid parameter. For the fluid phase, the velocity and temperature decreased in the increasing fluid-particles interaction parameter. On the contrary, the opposite trend is observed for the dust phase. The microrotation effect is more dominant near the wall since the micropolar fluid parameter increases, showing that the microrotation continuously decreases and becomes zero.

Acknowledgements

The authors would like to acknowledge The Ministry of Higher Education for Research Acculturation of Early Career Researchers (FRGS-RACER) (Ref: RACER/1/2019/STG06/UMP//1) through RDU192602 and Universiti Malaysia Pahang for the financial support through RDU182307. Deep appreciation also extended to Universiti Teknologi MARA Cawangan Pahang, Jengka Campus, for the guidance and support.

References

- [1] Saffman, P. G. "On the stability of laminar flow of a dusty gas." *Journal of fluid mechanics* 13, no. 1 (1962): 120-128. <https://doi.org/10.1017/S0022112062000555>
- [2] Aggarwal, Parul, and Urvashi Gupta. "Double-Diffusive Magnetoconvection of a Dusty Micropolar Fluid Saturating a Porous Medium." *International Scholarly Research Notices* 2013 (2013). <https://doi.org/10.1155/2013/789070>
- [3] Manjunatha, S., and B. J. Gireesha. "Effects of variable viscosity and thermal conductivity on MHD flow and heat transfer of a dusty fluid." *Ain Shams Engineering Journal* 7, no. 1 (2016): 505-515. <https://doi.org/10.1016/j.asej.2015.01.006>
- [4] Siddiqa, Sadia, M. Anwar Hossain, and Suvash C. Saha. "Two-phase natural convection flow of a dusty fluid." *International Journal of Numerical Methods for Heat & Fluid Flow* 25 (2015): 1542-1556. <https://doi.org/10.1108/HFF-09-2014-0278>
- [5] Izani, Siti Nur Haseela, and Anati Ali. "Hydromagnetic mixed convection flow over an exponentially stretching sheet with fluid-particle suspension." In *AIP Conference Proceedings*, vol. 1750, no. 1, p. 030043. AIP Publishing LLC, 2016. <https://doi.org/10.1063/1.4954579>
- [6] Siddiqa, Sadia, Naheed Begum, Md Anwar Hossain, and Rama Subba Reddy Gorla. "Natural convection flow of a two-phase dusty non-Newtonian fluid along a vertical surface." *International Journal of Heat and Mass Transfer* 113 (2017): 482-489. <https://doi.org/10.1016/j.ijheatmasstransfer.2017.05.080>
- [7] Arifin, N. S., S. M. Zokri, A. R. M. Kasim, M. Z. Salleh, N. F. Mohammad, and W. N. S. W. Yusoff. "Aligned magnetic field of two-phase mixed convection flow in dusty Casson fluid over a stretching sheet with Newtonian heating." In *Journal of Physics: Conference Series*, vol. 890, no. 1, p. 012001. IOP Publishing, 2017. <https://doi.org/10.1088/1742-6596/890/1/012001>
- [8] Hossain, Md Anwar, Nepal C. Roy, and Sadia Siddiqa. "Unsteady mixed convection dusty fluid flow past a vertical

- wedge due to small fluctuation in free stream and surface temperature." *Applied Mathematics and Computation* 293 (2017): 480-492. <https://doi.org/10.1016/j.amc.2016.08.048>
- [9] Kasim, Abdul Rahman Mohd, Nur Syamilah Arifin, Noor Amalina Nisa Ariffin, Mohd Zuki Salleh, and Muhammad Imran Anwar. "Mathematical model of simultaneous flow between Casson fluid and dust particle over a vertical stretching sheet." *International Journal of Integrated Engineering* 12, no. 3 (2020): 253-260.
- [10] Arifin, Nur Syamilah, Syazwani Mohd Zokri, Abdul Rahman Mohd Kasim, Mohd Zuki Salleh, and Nurul Farahain Mohammad. "Two-phase mixed convection flow of dusty Williamson fluid with aligned magnetic field over a vertical stretching sheet." In *Proceedings of the Third International Conference on Computing, Mathematics and Statistics (ICMS2017)*, pp. 209-216. Springer, Singapore, 2019. https://doi.org/10.1007/978-981-13-7279-7_26
- [11] Mohd Kasim, Abdul Rahman, Nur Syamilah Arifin, Syazwani Mohd Zokri, Mohd Zuki Salleh, Nurul Farahain Mohammad, Dennis Ling Chuan Ching, Sharidan Shafie, and Noor Amalina Nisa Ariffin. "Convective transport of fluid–solid interaction: A study between non-Newtonian Casson model with dust particles." *Crystals* 10, no. 9 (2020): 814. <https://doi.org/10.3390/cryst10090814>
- [12] Arifin, Nur Syamilah, Syazwani Mohd Zokri, Abdul Rahman Mohd Kasim, Mohd Zuki Salleh, and Noor Amalina Nisa Arifin. "Jeffrey fluid embedded with dust particles over a shrinking sheet: A numerical investigation." *Journal of Advanced Research in Fluid Mechanics and Thermal Sciences* 74, no. 2 (2020): 196-209. <https://doi.org/10.37934/arfmts.74.2.196209>
- [13] Kasim, Abdul Rahman Mohd, Nur Syamilah Arifin, Syazwani Mohd Zokri, and Mohd Zuki Salleh. "The investigation of a fluid-solid interaction mathematical model under combined convective jeffrey flow and radiation effect embedded newtonian heating as the thermal boundary condition over a vertical stretching sheet." In *Defect and Diffusion Forum*, vol. 399, pp. 65-75. Trans Tech Publications Ltd, 2020. <https://doi.org/10.4028/www.scientific.net/DDF.399.65>
- [14] Kasim, Abdul Rahman Mohd, Nur Syamilah Arifin, Syazwani Mohd Zokri, and Mohd Zuki Salleh. "Fluid-particle interaction with buoyancy forces on Jeffrey fluid with Newtonian heating." *CFD Letters* 11, no. 1 (2019): 1-16.
- [15] Dasman, A., Nur Syamilah Arifin, Abdul Rahman Mohd Kasim, and Nor Azizah Yacob. "Formulation of dusty micropolar fluid mathematical model." In *Journal of Physics: Conference Series*, vol. 1366, no. 1, p. 012032. IOP Publishing, 2019. <https://doi.org/10.1088/1742-6596/1366/1/012032>
- [16] Aljabali, Ahlam, Abdul Rahman Mohd D. Kasim, and Ayman Mohd Hussein. "A Progress on the Development of Mathematical Model on Two-Phase Flow over a Vertical Stretching Sheet." In *Journal of Physics: Conference Series*, vol. 1366, no. 1, p. 012045. IOP Publishing, 2019. <https://doi.org/10.1088/1742-6596/1366/1/012045>
- [17] Arifin, Nur Syamilah, Syazwani Mohd Zokri, Abdul Rahman Mohd Kasim, Mohd Zuki Salleh, Wan Nur Syahidah Wan Yusoff, Nurul Farahain Mohammad, and Sharidan Shafie. "Aligned magnetic field on dusty Casson fluid over a stretching sheet with Newtonian heating." *Malaysian Journal of Fundamental and Applied Sciences* 13, no. 3 (2017): 245-248. <https://doi.org/10.11113/mjfas.v13n3.592>
- [18] Qasim, Muhammad, Ilyas Khan, and Sharidan Shafie. "Heat transfer in a micropolar fluid over a stretching sheet with Newtonian heating." *Plos One* 8, no. 4 (2013): e59393. <https://doi.org/10.1371/journal.pone.0059393>
- [19] Turkyilmazoglu, M. "Flow of a micropolar fluid due to a porous stretching sheet and heat transfer." *International Journal of Non-Linear Mechanics* 83 (2016): 59-64. <https://doi.org/10.1016/j.ijnonlinmec.2016.04.004>