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Analytical Solution of Unsteady Casson Fluid Flow Through a Vertical Cylinder with Slip Velocity Effect

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ABSTRACT

Casson fluid is a non-Newtonian fluid with its unique fluid behaviour because it behaves like an elastic solid or liquid at a certain condition. Recently, there are several studies on unsteady Casson fluid flow through a cylindrical tube have been done by some researchers because it is related with the real-life applications such as blood flow in vessel tube, chemical and oil flow in pipelines and others. Therefore, the main purpose of the present study is to obtain analytical solutions for unsteady flow of Casson fluid pass through a cylinder with slip velocity effect at the boundary condition. Dimensional governing equations are converted into dimensionless forms by using the appropriate dimensionless variables. Dimensionless parameters are obtained through dimensionless process such as Casson fluid parameters. Then, the dimensionless equations of velocity with the associated initial and boundary conditions are solved by using Laplace transform with respect to time variable and finite Hankel transform of zero order with respect to the radial coordinate. Analytical solutions of velocity profile are obtained. The obtained analytical result for velocity is plotted graphically by using Maple software. Based on the obtained result, it can be observed that increasing in Casson parameter, time and slip velocity will lead to increment in fluid velocity. Lastly, Newtonian fluid velocity is uniform from the boundary to the center of cylinder while Casson fluid velocity is decreased when approaching to the center of cylinder. The present result is validated when the obtained analytical solution of velocity is compared with published result and found in a good agreement.

1. Introduction

Almost everything in the world is covered with fluids even the living of nature. Newtonian fluids are the fluids that obey the Newtonian law of viscosity. However, fluids which does not obey theory of Newtonian fluids are known as non-Newtonian fluid and it is attracted many researchers to study due to its wide applications by Yusof *et al.,* [1]. Casson fluid is a unique non-Newtonian fluid due to its plasticity behaviour. Casson fluid is defined as a shear-thinning fluid with yield stress which is to have an infinite viscosity at zero shear rate by Ullah *et al.,* [2]. Casson fluid flow in a cylindrical domain had been attracted many researchers to study a long time ago since it is more related to the real-life

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applications such as blood flow in vessel, oil flow in chemicals, and others. An example of early literature as early as 1973 which is studied Casson fluid flow through an infinite cylinder and solved it numerically by Shul'man [3].

Laplace and Hankel transform method are among the famous tools to solve the problems in cylindrical domain in order to obtain analytical solutions such as study had been done by Jamil *et al.,* [4] to solve the problem for blood flow in cylindrical tube. Recently, researchers have been studied analytical solutions of Casson fluid flow in cylinder by using Laplace and Hankel transform method. The earlier studies have been done by Ali *et al.,* [5-8] which are solved problems of Casson fluid flow in cylinder with the different type of boundary conditions for examples are fixed boundary conditions, constant velocity, and oscillating boundary conditions. Then, Maiti *et al.,* [9-10], and Imtiaz *et al.,* [11] discussed exact solutions of MHD Casson fluid flow with heat transfer effect in the cylinder with fixed boundary conditions. None of them solved the Casson fluid flow problem in cylinder with slip velocity as a boundary condition.

Nubar [12] defined slip as a finite velocity of a fluid at a boundary or surrounded within in that area. Most of the researchers solved slip problems which occurred on the plate. Nowadays, slip problems in cylinder attracted researchers and scientists to have further study related with it due to its applications used widely in the real life. Study towards the slip effects is significant since it is strongly influenced the flow velocity [13]. Hayat *et al.,* [14] analysed Casson fluid flow through a vertical stretching cylinder in the region of stagnation point with slip effects and it solved by using Homotopy Analysis. Then, the similar problem has been solved numerically by El-Aziz and Afify [15] with the additional effects of MHD and heat transfer by using shooting method with fourth-order Runge-Kutta. None of them solved slip velocity problem of Casson fluid flow in the cylinder by using Laplace and Hankel transform techniques.

In order to obtain analytical solutions for the slip velocity problem at the cylinder boundary, some researchers solved the problems by using Laplace and Hankel transform methods. For example, Jiang *et al.,* [16] solved Oldroyd-B fluid flow in a circular microchannel with the Navier slip velocity as a boundary condition. Then, the problem extended by Shah *et al.,* [17] with the additional of time fractional derivative model. Next, Padma *et al.,* [18-19] discussed the effects of slip velocity and without slip velocity for the Jeffrey fluid flow. However, researchers not yet solved the problem related with the Casson fluid model.

To sum up, the literature that had been discussed as above stated that no study has been reported yet in investigating Casson fluid flow analysis past through cylinder with slip velocity effect. Thus, the aim of the present work is to study on unsteady Casson fluid flow in a vertical cylinder with the slip velocity effect at the boundary. The analytical solutions are obtained by using the Laplace and finite Hankel transform methods. Laplace transform is one of the famous analytical techniques to solve problems in fluid mechanics which is involved with the time dependent and finite Hankel transform is one the best tool to solve problem which is related with the radial coordinate associated with initial and boundary value problems. Unsteady state involves with time dependent of initial and boundary value problems which is fitted with the proposed method to obtain exact solutions of governing equation.

2. Problem Formulation

Consider the flow of incompressible Casson fluid in an infinite vertical cylinder of radius, r_0. The z-axis is considered along the axis of cylinder in vertical upward direction and radial coordinate r is taken normal to it. The Casson fluid flow with slip velocity at the boundary is shown in Figure 1. Initially at time t^*=0, the fluid and cylinder are both at rest. Then, at the time t^* > 0, the fluid begins to flow due to the slip velocity, u_s is applied at the boundary of the cylinder and fluid has uniform flow velocity on the axis. Assume that the velocity are the function or *r* and *t* only. Then, under the usual Boussinesq's approximation, the corresponding partial differential equation for momentum is given as [10]

$$\rho \frac{\partial u^*}{\partial t^*} = \mu \left(1 + \frac{1}{\beta}\right)\left(\frac{\partial^2 u^*}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial u^*}{\partial r^*}\right) \tag{1}$$

with the associated initial and boundary conditions [18]

$$u^*(r^*,0) \quad =0 \quad ; \; r \in [0, r_0] \,, \tag{2}$$

$$u^*(r_0^*, t^*) \;\; = u_s^* \;\; ; \; t^* > 0. \tag{3}$$

where u^* is the velocity component along *z*-axis, μ is the dynamic viscosity of fluid, β is the non-Newtonian Casson parameter, ρ is the density of fluid, ν is the kinematic viscosity of fluid. Introducing the following dimensionless variables [18-20]

$$u_s = \frac{u_s^*}{u_0},\; u = \frac{u^*}{u_0},\; r = \frac{r^*}{r_0},\quad t = \frac{t^*\nu}{r_0^2} \tag{4}$$

and substitute Eq. (3) into Eq. (1) - (3), obtained dimensionless partial differential equations as

$$\frac{\partial u}{\partial t} = \beta_1\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \tag{5}$$

with the associated initial and boundary conditions

$$u(r,0) \;\; =0 \quad ; \; r \in [0,1] \,, \tag{6}$$

$$u(1,t) \;\; = u_s \;\; ; \; t > 0, \tag{7}$$

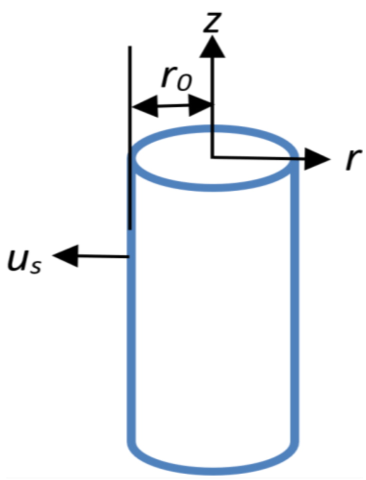

Fig. 1. Physical geometry of the fluid flow

where $\beta_1 = \frac{1}{\beta_0}$ and $\beta_0 = 1 + \frac{1}{\beta}$ are the constant parameters.

3. Problem Solution

The joint Laplace and Hankel transforms have been used to find analytical solutions of the Eq. (5). Apply Laplace transform into Eq. (5) and Eq. (7), and by using the initial condition (6), yields

$$s\bar{u}(r,s) = \beta_1 \left[\frac{\partial^2 \bar{u}(r,s)}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}(r,s)}{\partial r} \right], \tag{8}$$

$$\bar{u}(1,s) = \frac{u_s}{s}, \tag{9}$$

where $\bar{u}(r,s)$ is the Laplace transform of the function $u(r,t)$ and s is the transform variable. Then, apply finite Hankel transform of zero order to Eq. (8) and by using condition (9), give

$$\bar{u}_H(r_n,s) = \frac{J_1(r_n)}{r_n}\left[\frac{u_s}{s} - \frac{u_s}{s + {r_n}^2\beta_1}\right] \tag{10}$$

where $\bar{u}_H(r_n,s) = \int_0^1 r\bar{u}(r,s) J_0(r r_n)\, dr$ is the finite Hankel transform of the function $\bar{u}(r,s)$ and r_n, with $n=0,1,\ldots$ are the positive roots of the equation $J_0(x) = 0$, where J_0 is being the Bessel function of first kind and zero order. Next, the inverse Laplace transform of Eq. (10) is

$$u_H(r_n,t) = \frac{J_1(r_n)}{r_n} u_s - \frac{J_1(r_n)}{r_n} u_s\, exp(-{r_n}^2\beta_1 t). \tag{11}$$

Lastly, the inverse Hankel transform is applied to Eq. (11) and written as

$$u(r,t) = u_s - 2u_s \sum_{n=1}^{\infty} \frac{J_0(r r_n)}{r_n J_1(r_n)} exp(-{r_n}^2\beta_1 t). \tag{12}$$

3. Result and Discussion

The flow information is obtained from numerical result by using a Maple code for the analytical solution Eq. (12). In order to verify the accuracy of the present result Eq. (12), the limiting case of the present result is compared with the published result [20]. This comparison is shown in Figure 2. It is found that limiting result Eq. (12) is identical to Eq. (29) obtained by Khan *et al.,* [20]. This confirmed the accuracy of the obtained result.

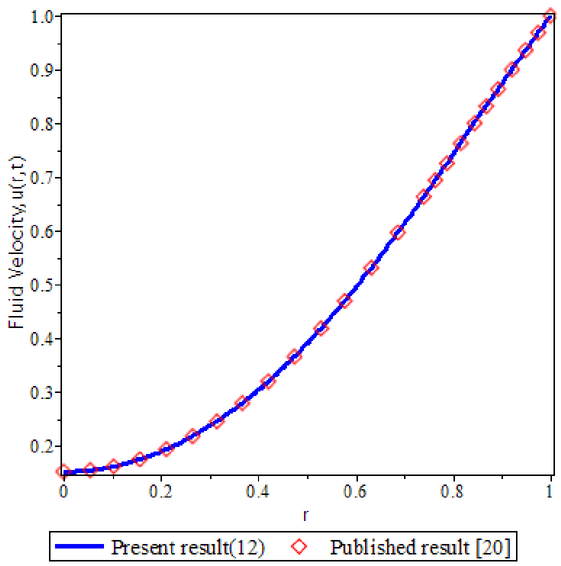

Fig. 2. Comparison of the present result Eq. (12) when $\beta \to \infty$, u_s=1.0 with the published result when *Gr* =*ω*= 0 [20]

The impact of various fluid parameters on the fluid velocity *u*(*r,t*) versus radial coordinate *r* has been discussed graphically in Figure 3 to Figure 6. The following parametric values for numerical computation have been estimated based on the physical values provided in Padma *et al.,* [18-19] as follows: u_s=1.0, *t*=1.0. However, to characterize the results of present study, wide spectrum values of parameters had been used as *β*=0.1,1.0,3.0; *t*=0.1,2.0,4.0; u_s=0.1,0.2,0.3. Meanwhile, *β*=0.1 is chosen to show the effect of its viscosity.

The influence of Casson parameter *β* on velocity profile is exhibited in Figure 3. From the observation, fluid velocity increases as Casson parameter increases. It is due to the fact that when Casson parameter increases it will cause yield stress to fall and the boundary layer thickness decreases which is resulting in the enhancement of the velocity.

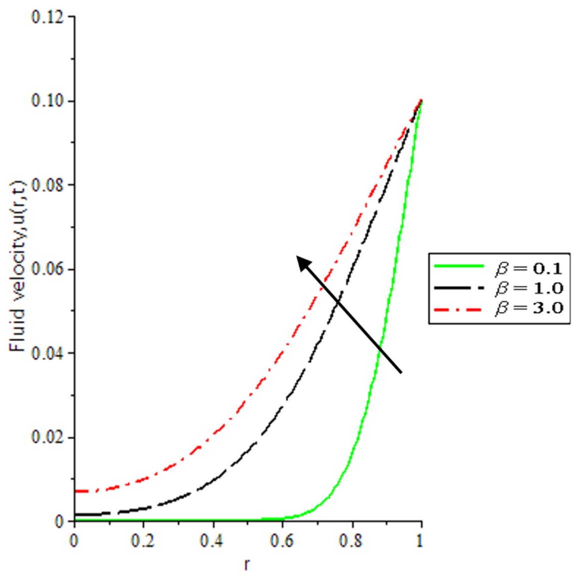

Fig. 3. Velocity profiles for different values of Casson parameters with *t*=1.0, u_s=0.1

Figure 4 displays the velocity profiles with the time changes. It is found that fluid velocity is increased when the value of time *t* is increased. Besides, it can be observed that fluid velocity at the boundary and approaching to the center of cylinder will be same as time *t* is increased. It is due to the slip velocity is applied at the boundary of cylinder which results in increasing fluid particle movement when time is increasing and lead to increase of fluid velocity. Consequently, fluid velocity will achieve uniform velocity in the cylinder as time *t* is increased.

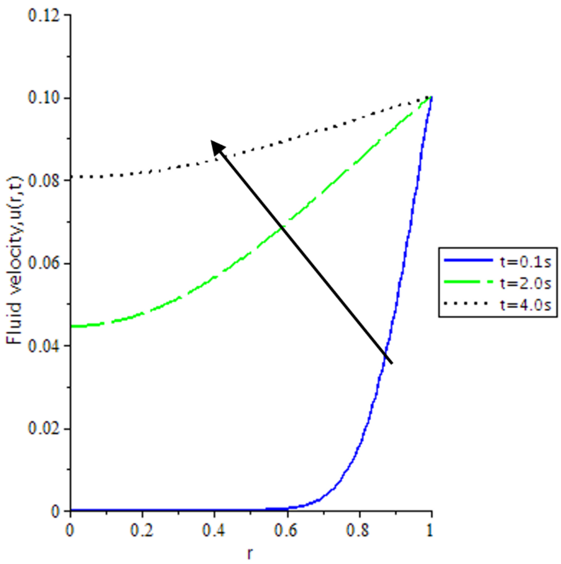

Fig. 4. Velocity profiles for different values of time with β=0.1, u_s=0.1

The diagram of Figure 5 is plotted in order to discuss the influence of slip velocity at the boundary on fluid velocity. The trend of the graph shows the different of fluid velocity is occurred from the cylinder wall surface to the center of cylinder. Fluid velocity at the cylinder surface is equal to the applied slip velocity at the boundary. Meanwhile, fluid velocity is decreased as approaching to the center of cylinder. It is due to the boundary layer thickness of Casson fluid since Casson parameter β=0.1 is chosen in this graph. In the other word, fluid velocity will decrease as it is approaching to the center due to the boundary layer thickness of Casson fluid is larger.

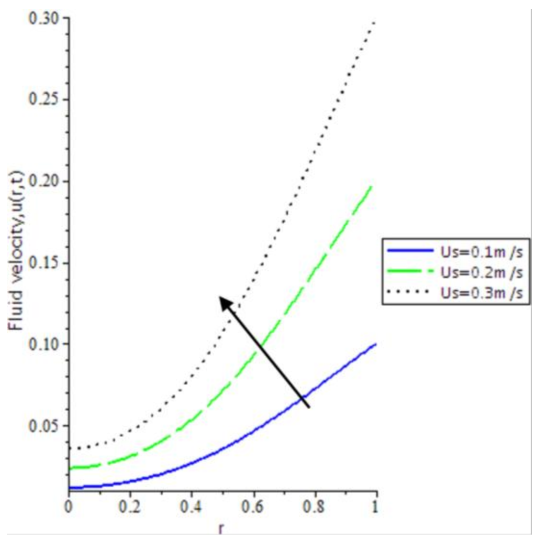

Fig. 5. Velocity profiles for different values of slip velocity with β=0.1, t=1.0

Figure 6 illustrates the effect of slip velocity at the boundary on the fluid velocity between Casson fluid and Newtonian fluid. When $\beta \to \infty$, the yield stress of Casson parameter will approach to zero and the flow will behave as a Newtonian fluid while when $\beta \to 0$, the non-Newtonian characteristics become more effective. It can be observed that fluid velocity of Newtonian fluid is uniform from the boundary to the center of cylinder meanwhile fluid velocity of Casson fluid decreases as it is approaching to the center of cylinder. The reason is the velocity boundary layer thickness for Casson fluid is larger than those for Newtonian fluid due to Casson fluid unique behavior which is its plasticity behaviour.

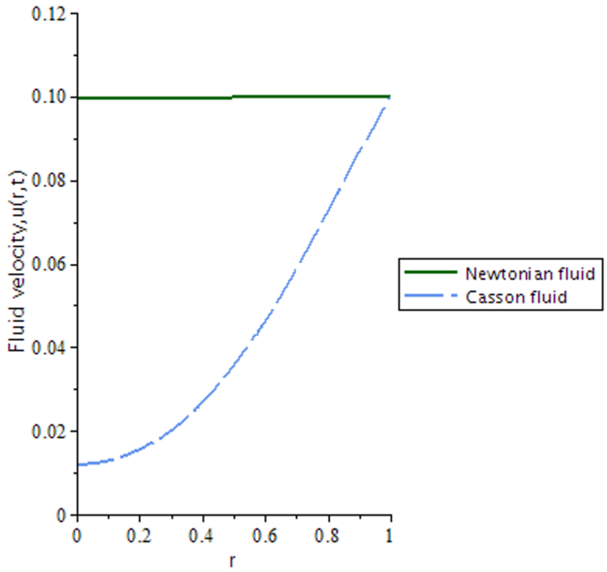

Fig. 6. Velocity profiles of Newtonian fluid (β=∞) and Casson fluid (β=0.1) with u_s=0.1, t=1.0

5. Conclusion

The problem of unsteady Casson fluid flow through a vertical cylinder with the slip velocity effect at the boundary is studied. The Laplace and finite Hankel transform methods had been used to attain analytical solutions for this problem. Finally, the solutions are satisfactory with related initial and boundary conditions. The obtained solutions are discussed graphically with the effects of Casson parameter β, time parameter t, and slip velocity parameter u_s. The velocity profiles increase when t, β, and u_s increase. Besides that, the fluid velocity for Newtonian fluid is uniform from the boundary to the center of cylinder while Casson fluid velocity is decreased when approaching to the center of cylinder.

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