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Simulation of Internal Undular Bores in Rapidly Varying Topography

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ABSTRACT

This paper intends to look at the influence of rapidly varying regions on the propagation of internal undular bores in a two-layer fluid flow. Internal undular bores have been observed in the ocean around the world. The appropriate mathematical model to describe the evolution of internal undular bores in a stratified fluid is the variable-coefficient extended Korteweg-de Vries equation. The governing equation is solved numerically using the method of lines to simulate the propagation of internal undular bores. Our numerical results show that the effects of rapidly varying topography lead to adiabatic and non-adiabatic deformation of the internal undular bores, including the generation of solitary wavetrain, generation of nonlinear wavetrain and generation of rarefaction wave. Besides, multi-phase behaviour is also observed during the evolution.

1. Introduction

Internal undular bores have been observed propagating in coastal regions, e.g. Australian North West Shelf [1], Andaman Sea [2], etc. due to fluid stratification [3-4]. Undular bore connects two different basic flow states and has a structure of slowly modulated nonlinear periodic waves with soliton at the leading edge [5]. In a real-world situation, the bottom topography of the oceans is not constant and varies most of the time. Hence, the propagation of the internal waves is often under the influence of variable topography and Earth's rotation [6-7].

The appropriate model for internal waves is the variable-coefficient extended Korteweg-de Vries (veKdV) equation since very often internal waves have large amplitudes. Grimshaw et al. [8] (also see references therein) gave a detailed derivation of the veKdV equation for internal waves. When a single isolated internal solitary wave propagates over a slowly varying topography, it would exhibit several adiabatic and non-adiabatic effects due to the topographic effects. These effects include the formation of a trailing shelf which would then decomposes into a secondary wavetrain, the generation of an internal solitary wave of opposite polarity, etc. [3-4, 9-11]. Furthermore, internal solitary waves will evolve or fission into several solitons when they propagate over a step or rapidly

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changing bottom, i.e. when the coefficients of the nonlinearity and dispersion terms in the veKdV equation change rapidly with respect to the wavelength of a solitary wave [8, 12-13].

Similarly, the evolution of internal undular bores over slowly varying depth regions would also lead to several adiabatic and non-adiabatic effects including the generation of a solitary wavetrain ahead of the transformed internal undular bore, the generation of a step-like wave, etc. [15-16]. Our aim of this work is to simulate the evolution of the internal undular bores over rapidly varying depth regions and study how the variable topography affects the internal undular bores, particularly the leading solitary wave at the leading edge. In the next section, we would present the formulation of the problem followed by numerical results and discussions in the new section. The final section is the conclusion.

2. Problem Formulation

The propagation of internal waves can be described by the following veKdV equation (1). A detailed derivation has been given by Grimshaw *et al.,* [4, 8].

$$U_t + \alpha U U_x + \beta U^2 U_x + \gamma U_{xxx} = 0 \tag{1}$$

The coefficients α, β and γ are described respectively as follows

$$\left.\begin{aligned}
\alpha &= \frac{3Qc\left(\rho_2 H_1^2 - \rho_1 H_2^2\right)}{2H_1 H_2\left(\rho_2 H_1 + \rho_1 H_2\right)} \\
\beta &= \frac{-3cQ^2\left[\left(\rho_1 H_2^2 - \rho_2 H_1^2\right)^2 + 8\rho_1\rho_2 H_1 H_2\left(H_1 + H_2\right)^2\right]}{8H_1^2 H_2^2\left(\rho_2 H_2 + \rho_2 H_1\right)^2} \\
\gamma &= \frac{cH_1 H_2\left(\rho_1 H_1 + \rho_2 H_2\right)}{6\left(\rho_1 H_2 + \rho_2 H_1\right)}
\end{aligned}\right\} \tag{2}$$

where

$$Q^2 = \frac{1}{2g\left(\rho_2 - \rho_1\right)c} \tag{3}$$

and

$$c = \sqrt{\frac{g\left(\rho_2 - \rho_1\right)H_1 H_2}{2\rho_1 H_2}} \tag{4}$$

Here U is the wave amplitude. The depth of the top and bottom layers are denoted by H_1 and H_2 respectively, while ρ_1 and ρ_2 denote the densities of both layers. We shall suppose that the top layer has constant depth, i.e. $H_1 = 1.5$ and the depth of the bottom layer changes instantly from $H_2 = h_0$ to $H_2 = h_1$ at $T = T_0$. Figure 1 illustrates the schematic of the problem.

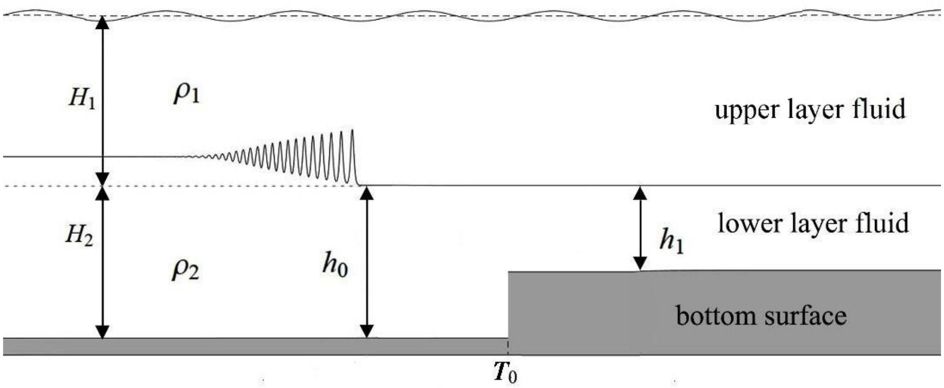

Fig. 1. Schematic illustration of the problem

The initial condition is taken as

$$U_0 = \frac{b}{2}\left[1 - \tanh\left(\frac{X}{20}\right)\right] \tag{5}$$

where b is the height of the sharp step between two regions of different constant depths. We consider b = 0.15 and the initial condition is placed far before the slope such that the undular bore is fully developed (refer to Figure 2 below) before it enters the new constant depth region.

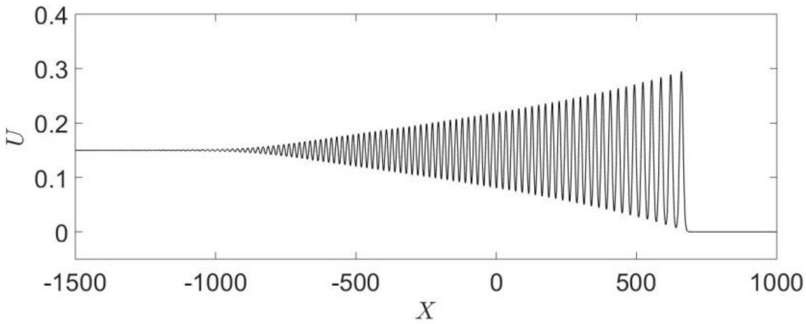

Fig. 2. A fully developed undular bore

Eq. (1) is solved using the method of lines, which has been used to solve many nonlinear wave equations [17-19] to simulate the evolution of the internal undular bore over a rapidly changing slope.

3. Results

We shall present all the numerical results in this section. There are two different types of rapidly changing topography considered in this paper, i.e. rapidly decreasing depth and rapidly increasing depth.

3.1 Rapidly Decreasing Depth Region

First, we let the lower layer has the following bottom profile

$$H_2(T) = \begin{cases} 1.0, & 0 \le T < 100, \\ 0.7, & T \ge 100. \end{cases} \tag{6}$$

Figure 3 shows the evolution of an internal undular bore over a rapidly decreasing depth region. As the leading solitary wave of the internal undular bore enters the new constant depth region, it behaves similarly to an isolated single internal solitary wave and fission into a solitary wavetrain, i.e. disintegrate into a few smaller solitary waves. The smaller waves of the solitary wavetrain would then interact with the remainder of the undular bore and cause the multi-phase interaction region. Once the multi-phase interaction is allowed to settle down over time, the transformed bore on the new constant depth region holds its structure, i.e. single-phase structure. One can observe that there is a sequence of soliton formed ahead of the transformed bore. The amplitude of the leading wave of the transformed undular bore remains as 2*b* on the new region.

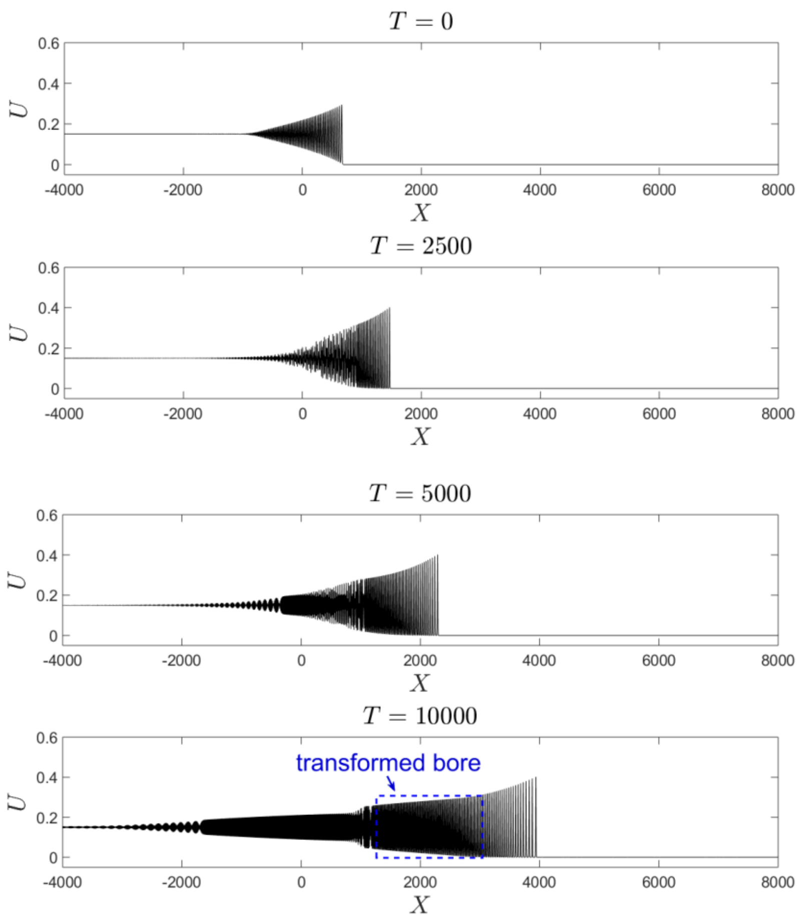

Fig. 3. Propagation of internal undular bore over rapidly decreasing depth region

Figure 4 shows the amplitude variations of the leading solitary wave of the initial undular bore and the isolated single solitary wave. One can observe that the leading solitary wave of the internal undular bore behaves exactly as the isolated single internal solitary wave and deforms adiabatically as described by weak soliton interaction at the leading edge.

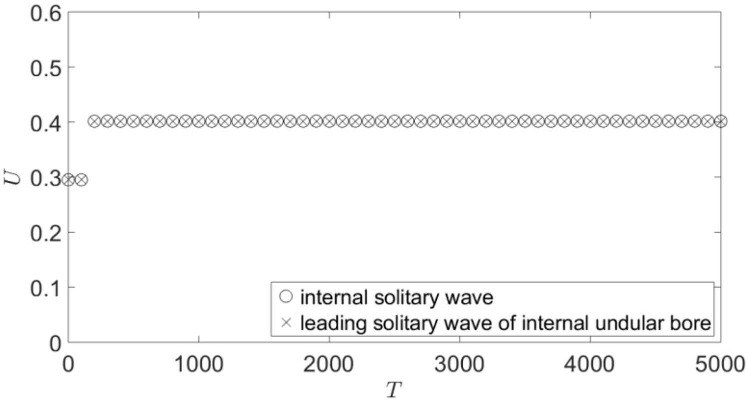

Fig. 4. Amplitude variation of leading solitary wave of internal undular bore and isolated single internal solitary wave

3.2 Rapidly Increasing Depth Region

Now, we let the profile of the lower layer is described by

$$H_2(T) = \begin{cases} 1.0, & 0 \le T < 100, \\ 1.3, & T \ge 100. \end{cases} \tag{7}$$

Here, the internal undular bore is propagating over a rapidly increasing depth region, where the lower layer on the new constant region has a smaller or equal depth to the upper layer, $h_1 \le H_1$. In this case, there is no polarity change in the internal undular bores, which is determined by the value of α in Eq (1).

As the leading solitary wave reaches the new depth region, the amplitude of the leading solitary wave of the internal undular bore decreases immediately (see Figure 5 when T = 200). This causes the formation of multi-phase interaction in the evolving wave structure. The strong soliton interaction is observed at the leading edge, i.e. the evolution of the leading wave is not only determined by the local variation due to the varying topography but also by the interaction between the leading solitary wave and the nonlinear wavetrain behind it. This prevents the amplitude of the leading wave continue to decline and causes the leading wave to grow until it reaches a new limiting amplitude, $U_{\lim} \approx 0.2057$ (see Figure 5 when T = 2500 to T = 50000). At larger time-scale propagation, the transformed internal undular bore contains two distinct wave structures, i.e. the transformed undular bore at the front part of the bore connected by a weakly nonlinear wave structure which is a part of the initial bore.

Figure 6 displays the 3D plot of the evolution of the internal undular bore over the rapidly increasing depth region. Figure 7 shows the comparison of the amplitude variation between the leading solitary wave of the internal undular bore and the isolated single internal solitary wave. The amplitude of the single solitary wave decreases rapidly on the new constant depth region. However, the amplitude of the leading solitary wave of the internal undular bore decreases and increases again due to the strong soliton interaction near the leading edge.

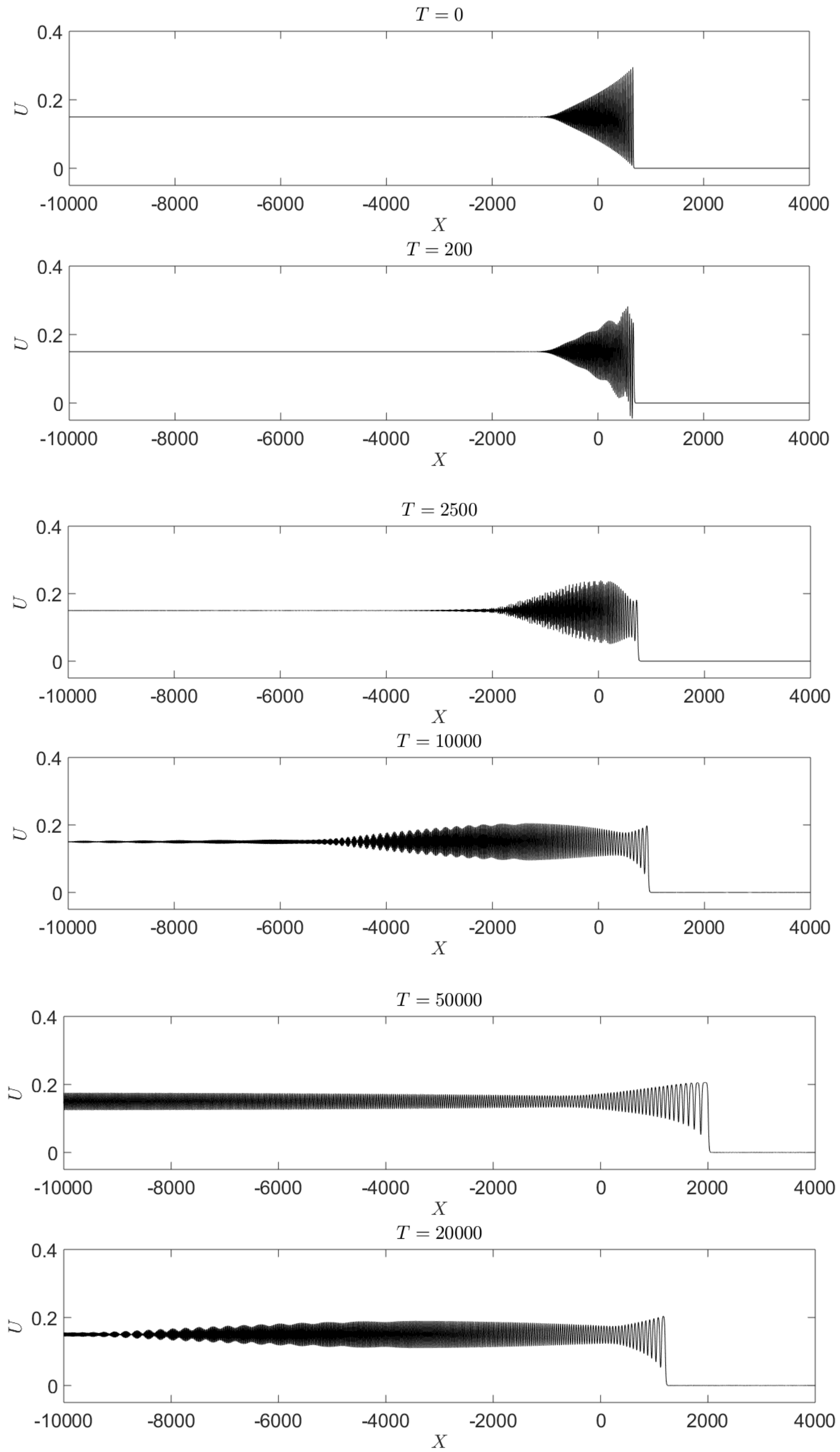

Fig. 5. Propagation of internal undular bore over rapidly increasing depth region

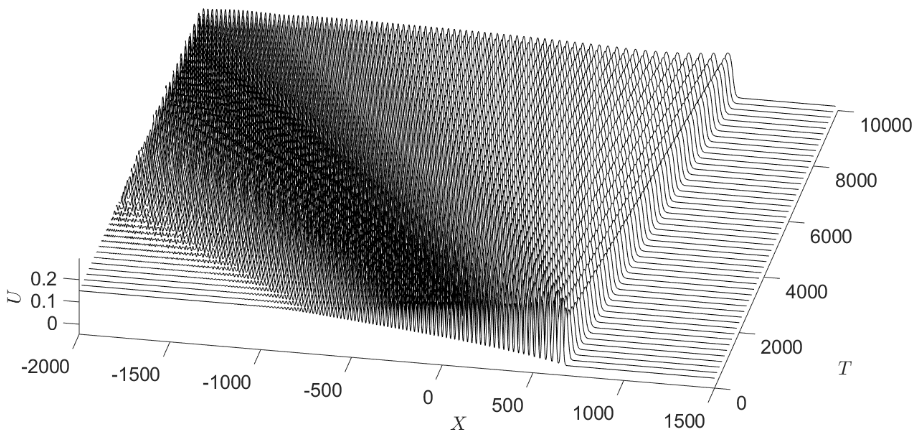

Fig. 6. 3D-plot of the propagation of internal undular bore over rapidly increasing depth region

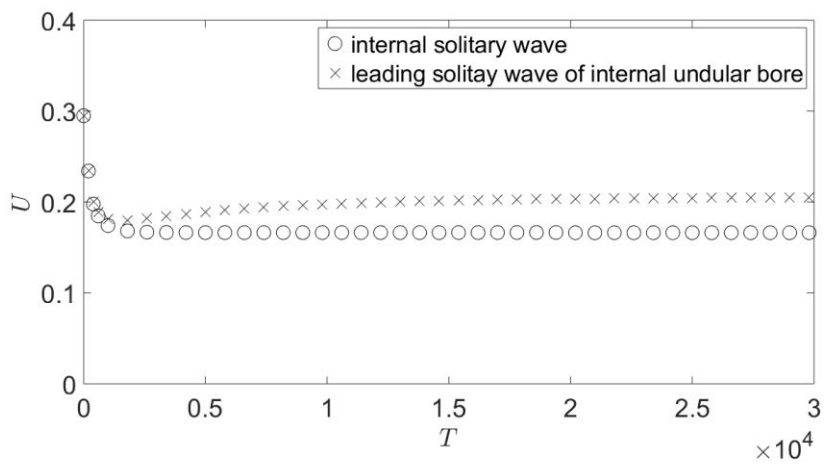

Fig. 7. Amplitude variation of the leading solitary wave of internal undular bore and isolated single internal solitary wave

Next, we look at the propagation of internal undular bores over a rapidly increasing depth region where the lower layer fluid on the new region has a deeper depth than the upper layer, $h_1 > H_1$. The depth profile of the lower layer varies as follows

$$H_2(T) = \begin{cases} 1.0, & 0 \le T < 100, \\ 1.7, & T \ge 100. \end{cases} \tag{8}$$

In this case, the evolution of the internal undular bore involves polarity change. As shown in Figure 8, the amplitude of the leading wave decreases immediately and the polarity of the wave changes from positive to negative when they suffer a sudden increase in the depth of lower layer fluid. The polarity change leads to the formation of a rarefaction wave at the front part followed by a transformed bore of negative polarity. In addition, temporal multi-phase interaction is evident to be seen throughout the evolution. As the transformed bore continues its propagation on the new constant depth region, some solitary waves of negative polarity separate at the leading edges of the transformed bore and propagate on the positive pedestal. Hence, a formation of a solitary wavetrain of negative polarity climbing on the positive pedestal is observed. However, the amplitude of these solitary waves slowly decreases and they would diminish on the pedestal at a larger time scale. Furthermore, the whole structure of the transformed bore is diminishing as time increases.

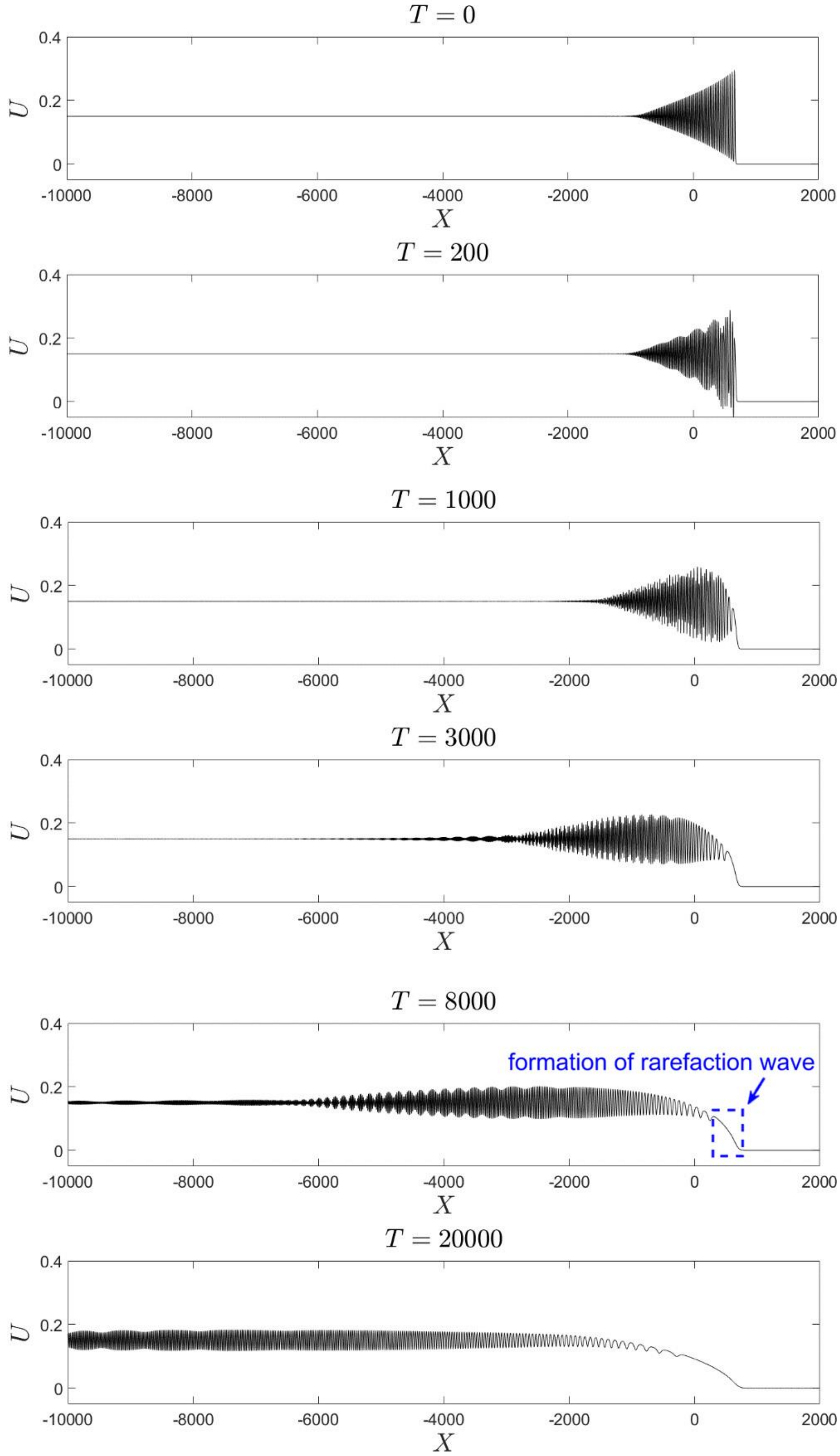

Fig. 8. Propagation of internal undular bore over rapidly increasing depth region

Figure 9 shows the 3D plot of the evolution of the internal undular bore. One can see the formation of a rarefaction wave at the front of the bore clearly from the 3D plot.

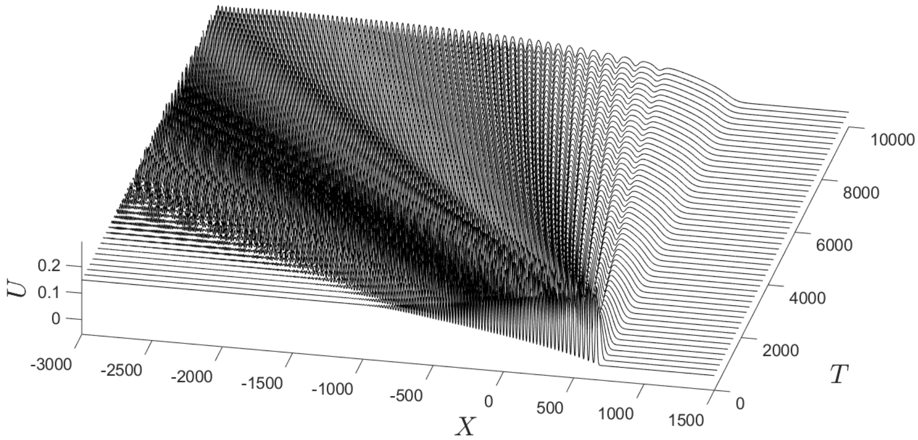

Fig. 9. Amplitude variation of leading solitary wave of internal undular bore and isolated single internal solitary wave

4. Conclusions

We have looked at the evolution of the internal undular bore over a rapidly changing bottom topography in this paper. From the numerical results, the evolution of the internal undular bore involves some interesting observations including, the formation of a solitary wavetrain ahead of the transformed bore, the generation of a new bore followed by an extending weakly nonlinear wavetrain at the rear part, the occurrence of temporary multi-phase behaviour, formation of rarefaction wave and diminishing transformed undular bore due to polarity change. These insights can be taken into account during the construction of coastal structures near coastal areas to reduce the impact of the waves.

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