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The Comparison Study of the Hybrid Method for Solving the Unsteady State Two-Dimensional Convection-Diffusion Equations

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ABSTRACT

In this paper, we present a hybrid method combining the reduced differential transform method (RDTM) and a resumption method based on Yang transform and Padé approximant to find analytical solutions for three test problems for the unsteady state two-dimensional convection-diffusion equation. The proposed method significantly improves the approximate solution series and broadens the convergence field of RDTM. The numerical results obtained are compared to RDTM and other results from previous works. The results show that the proposed method is very efficient and has high accuracy. The main advantage of the proposed method is that it is based on a few straightforward steps and does not generate secular terms or depend on a perturbation parameter. We also provided a powerful and attractive mathematical tool for solving linear and nonlinear equations.

1. Introduction

This work is interested in two unsteady state two-dimensional initial-boundary value problems, which were formulated as follows

Problem-I: Linear transport (convection-diffusion) equation:

$$\frac{\partial u}{\partial t}+\beta_x\frac{\partial u}{\partial x}+\beta_y\frac{\partial u}{\partial y}-\alpha_x\frac{\partial^2 u}{\partial x^2}-\alpha_y\frac{\partial^2 u}{\partial y^2}=0,\,(x,y,t)\in[0,L]\times[0,L]\times[0,T] \tag{1}$$

with initial condition

$$u(x,y,0)=\emptyset_o(x,y), 0\leq x,y\leq L, \tag{2}$$

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and the boundary conditions

$$\left.\begin{array}{l} u(x,0,t) = f_o(x,t), u(x,L,t) = f_1(x,t), 0 \le x \le L, t \ge 0 \\ u(0,y,t) = g_o(y,t), u(L,y,t) = g_1(y,t), 0 \le y \le L, t \ge 0 \end{array}\right\}, \tag{3}$$

where $u(x,y,t)$ is a transported variable β_x and β_y, are arbitrary constants which show the speed of convection and diffusion coefficients α_x and α_y are positive constants and $f_{0,}\, f_{1,}\, g_{0,}\, g_1$ and $\emptyset_o$ are known functions.

Problem-II: Nonlinear Burgers equation:

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + \alpha v \frac{\partial u}{\partial y} - \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0, \tag{4}$$

$(x,y,t) \in [0,L] \times [0,L] \times [0,T]$,

$$\frac{\partial v}{\partial t} + \alpha u \frac{\partial v}{\partial x} + \alpha v \frac{\partial v}{\partial y} - \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = 0, \tag{5}$$

with initial conditions

$$\left.\begin{array}{l} u(x,y,0) = a_1(x,y) \\ v(x,y,0) = a_2(x,y) \end{array}\right\} (x,y) \in \Omega,$$

and Dirichlet boundary conditions

$$\left.\begin{array}{l} u(x,y,t) = b_1(x,y,t), \\ v(x,y,t) = b_2(x,y,t), \end{array}\right\} (x,y) \in \Omega, t > 0,$$

where $\Omega = \{(x,y): a \le x \le b, c \le y \le d\}$ is the computational domain and $\partial\Omega$ is its boundary, $u(x,y,t)$ and $v(x,y,t)$ are the velocity components to be determined, a_1, a_2, b_1 and b_2 are the known functions, u_t is unsteady term, uu_x is the nonlinear convection term, $\frac{1}{Re}\left(u_{xx} + u_{yy}\right)$ is the diffusion term, $u_t = \frac{\partial u}{\partial t}, v_t = \frac{\partial v}{\partial t}, u_x = \frac{\partial u}{\partial x}, u_y = \frac{\partial u}{\partial y}, v_x = \frac{\partial v}{\partial x}, v_y = \frac{\partial v}{\partial y}, u_{xx} = \frac{\partial^2 u}{\partial x^2}, u_{yy} = \frac{\partial^2 u}{\partial y^2}, v_{xx} = \frac{\partial^2 v}{\partial x^2}, v_{yy} = \frac{\partial^2 v}{\partial y^2}$, and Re is the Reynolds number.

These problems are essential examples of partial differential equations that are represent a wide range of phenomena such as heat transfer, mass transfer, petroleum reservoir modeling, subsurface pollution remediation, continuum mechanics, shock waves, acoustic waves, gas dynamics, elasticity, and so on [1-4]. There is a wide body of literature on many forms of transport equations that are solved using various numerical and analytical approaches, for example: Tanaka and Chen [5] studied coupling dual reciprocity boundary element method and differential quadrature method for time-dependent diffusion problems. Bahadır [6] applied fully implicit finite-difference scheme for solving two-dimensional Burgers equations. Al-Saif and Al-Kanani [7] suggested alternative direction implicit formulation of the differential quadrature method for solving the unsteady state two-dimensional convection-diffusion equations. Abdou and Soliman [8] used variational iteration method for solving Burger's and coupled Burger's equations. Djidjeli *et al.,* [9] studied global and compact meshless schemes for the unsteady convection-diffusion equation. Sharma and Methi [10] presented homotopy perturbation method approach for the solution of equation to unsteady flow of a polytropic gas. You [11] proposed a high-order pade' ADI method for solving unsteady convection-diffusion equations. There are several, modern analytical methods, for example, reduced differential transform method (RDTM), was suggested by the Turkish mathematician Keskin and Oturanc [12] for the first time in 2009. It has received much attention since it has been applied to solve a wide variety of problems by many authors [13-18]. The Yang transform (YT) is suggested by Yang [19] in 2016, which is applied to solve the steady heat transfer problem, and it is used by several researchers in [20,21]. Henri Padé (1863-1953) presented an approximation technique in his doctoral thesis in 1892 which is called Padé approximation. It has received much attention since it has been applied to solve a wide variety of problems by many authors [22-25].

Therefore, in this study, we present a hybrid method that combines (RDTM, Padé approximation, and Yang transform) to find analytical solutions for PDE. The summary of the suggested method is that the solution to PDE is obtained in convergent series forms using YRDTM. After that, we create its Padé approximate function in order $[L/M]$ to convert the power series solution obtained by YRDTM into a meromorphic function. The values for L and M are selected at random. At this stage, the Padé approximant improves the accuracy and convergence of the truncated series solution by expanding the domain of that solution. This is a method that we call PYRDTM. Furthermore, unlike other semi-analytical methods such as homotopy perturbation method (HPM) and homotopy analysis method (HAM), or variational iteration method (VIM), Padé-Yang reduced differential transform method (PYRDTM) does not require a perturbation parameter to work and does not generate secular terms (noise terms). To solve three test problems, we presented the PYRDTM as a handy tool with great potential to solve linear or nonlinear unsteady state two-dimensional convection-diffusion equations and compared its efficiency and accuracy with other methods such as discrete Adomian decomposition method (DADM) and Bahadir. The numerical results we obtained showed the efficiency, activity, and accuracy of PYRDTM for solving the unsteady-state two-dimensional convection-diffusion equation.

2. Yang Transform

The integral transforms play a significant role in a variety of scientific disciplines and works of literature; they are used extensively in mathematical physics, optics, mathematical engineering, and other disciplines to solve differential equations such as those Mellin, Hankel, Sumudu, Laplace, and Fourier [26,27]. Recently, Yang [19] proposed a new integral transform named the Yang Transform, which was first applied to the steady-state heat transfer equation.

Definition 2.1
A new integral transform called Yang transform of the function $u(t)$ is denoted by $Y\{u(t)\}$ or $T(s)$ and is defined as [19]

$$Y[u(t)] = T(s) = \int_0^\infty e^{\frac{-t}{s}} u(t)dt, t > 0, \quad (7)$$

Provided the integral exists for some s, where $s \in (-t_1, t_2)$.

If we substitute $\frac{-t}{s} = x$ then Eq. (7) becomes,

$$Y[u(t)] = T(s) = s\int_0^\infty e^{-x} u(sx)dx, x > 0. \quad (8)$$

Theorem (1): Yang Transform of Derivatives [19]

If $Y[u(t)] = T(s)$ then

i. $Y\left[\frac{\partial u(x,t)}{\partial t}\right] = \frac{T(x,s)}{s} - u(x,0),$

ii. $Y\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right] = \frac{T(x,s)}{s^2} - \frac{u(x,0)}{s} - \frac{\partial u(x,0)}{\partial t},$

iii. $Y\left[\frac{\partial^n u(x,t)}{\partial t^n}\right] = \frac{T(x,s)}{s^n} - \sum_{k=0}^{n-1} s^{-n+k+1} \frac{\partial^{(k)} u(x,0)}{\partial t^k} \ \forall n = 1,2,3,4, \ldots$

2.1 Some Functions' Yang Transform

Yang transform of some useful functions is given below.

i. $Y\{1\} = s$

ii. $Y\{t\} = s^2$

iii. $Y\{t^n\} = n! \cdot s^{n+1}$

iv. $Y\{e^{at}\} = \frac{s}{1-as}$

v. $Y\{\sin\ at\} = \frac{as^2}{1+a^2s^2}$

vi. $Y\{\cos\ at\} = \frac{s}{1+a^2s^2}$

vii. $Y\{\sin\ at\} = \frac{as^2}{1-a^2s^2}$

viii. $Y\{\cos\ at\} = \frac{s}{1-a^2s^2}$.

3. Reduced Differential Transform Method

In 2009, the Turkish mathematician Keskin and Oturanc [12] suggested the reduced differential transform method (RDTM) to study the analytical solutions of the linear and nonlinear wave equation. The basic definitions and operations of two-dimensional reduced differential transform method are introduced as follows [14,18].

Definition 3.1
If function $u(x, y, t)$ is analytic and differentiated continuously with respect to time and space in the domain of interest, then let

$$U_k(x, y) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x, y, t)\right]_{t=0},$$

where, the t-dimensional spectrum function $U_k\ (x, y)$ is the transformed function. In this paper, the lower case $u(x, y, t)$ represents the original function, while the upper case $U_k\ (x, y)$stand for the transformed function.

Definition 3.2
The differential inverse transform of $U_k\ (x, y)$ is defined as;

$$u(x, y, t) = \sum_{k=0}^{\infty} U_k(x, y)(t - t_0)^k\ , \tag{10}$$

then by combining Eq. (9) and Eq. (10), we obtain

$$u(x,y,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,y,t)\right]_{t=0} t^k. \tag{11}$$

Note that the function $u(x,y,t)$ can be written in a finite series as follows:

$$u_n(x,y,t) = \sum_{k=0}^{\infty} U_k(x,y)(t-t_0)^k + R_n(x,y,t). \tag{12}$$

Where the tail function $R_n(x,y,t)$, is negligibly small. Therefore, the exact solution of problem is given $u(x,y,t) = \lim_{n\to\infty} u_n(x,y,t)$.

Table 1
The fundamental operations of RDTM

Functional Form	Transformed Form
$u(x,y,t)$	$U_k(x,y) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,y,t)\right]_{t=0}$
$w(x,y,t) = \alpha u(x,y,t) \pm \beta v(x,y,t)$	$W_k(x,y) = \alpha U_k(x,y) \pm \beta V_k(x,y)$
$w(x,y,t) = u(x,y,t)v(x,y,t)$	$W_k(x,y) = \sum_{r=0}^{\infty} V_r(x,y)U_{k-r}(x,y)$ $= \sum_{r=0}^{\infty} U_r(x,y)V_{k-r}(x,y)$
$w(x,y,t) = \frac{\partial^r}{\partial t^r} u(x,y,t)$	$W_k(x,y) = \frac{(k+r)!}{k!} U_{k+r}(x,y)$
$w(x,y,t) = \frac{\partial^2}{\partial y^2} u(x,y,t)$	$W_k(x,y) = \frac{\partial^2}{\partial y^2} U_k(x,y)$
$w(x,y,t) = \frac{\partial^2}{\partial x^2} u(x,y,t)$	$W_k(x,y) = \frac{\partial^2}{\partial x^2} U_k(x,y)$

4. The Padé Approximants

Suppose that, we are given a power series $\sum_{i=0}^{\infty} c_i x^i$, representing a function $f(x)$, so that

$$f(x) = \sum_{i=0}^{\infty} c_i x^i, \tag{13}$$

The Padé approximant is a rational fraction and the notation for such the Padé approximant is [22]

$$[L/M] = \frac{A_L(x)}{B_M(x)}, \tag{14}$$

where $A_L(x)$ is a polynomial of degree at most L and $B_M(x)$ is a polynomial of degree at most M. We have

$$\begin{aligned} f(x) &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots, \\ A_L(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_L x^L, \\ B_M(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots + b_M x^M \end{aligned} \tag{15}$$

Notice that in Eq. (14) there are $L+1$ numerator coefficients and $M+1$ denominator coefficients. Since we can clearly multiply the numerator and denominator by a constant and leave $[L, M]$ unchanged, we impose the normalization condition

$$B_M(0) = 1 = b_0. \tag{16}$$

So, there are $L + 1$ independent numerator coefficients and M independent denominator coefficients, making $L + M + 1$ unknown coefficients in all. This number suggests that normally the $[L, M]$ ought to fit the power series (13) through the orders $1, x, x^2, \dots, x^{L+M}$. By using the conclusion given by Baker and Graves-Morris [22], we know that the $[L, M]$ approximant is uniquely determined.

In the notation of formal power series,

$$\sum_{i=0}^{\infty} c_i x^i = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_L x^L}{1 + b_1 x + b_2 x^2 + \cdots + b_M x^M} + \mathrm{o}(x^{L+M+1}). \tag{17}$$

By cross-multiplying Eq. (17), we find that

$$(1 + b_1 x + b_2 x^2 + \cdots + b_M x^M)(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots) = a_0 + a_1 x + a_2 x^2 + \cdots + a_L x^L + \mathrm{o}(x^{L+M+1}) . \tag{18}$$

From Eq. (18), the set of equations can be found.

$$\begin{cases} c_0 = a_0, \\ c_1 + c_0 b_1 = a_1, \\ c_2 + c_1 b_1 + c_0 b_2 = a_2, \\ \vdots \\ c_L + a_{L-1} b_1 + \cdots + c_0 b_L = a_L, \end{cases} \tag{19}$$

and

$$\begin{cases} c_{L+1} + c_L b_1 + \cdots + c_{L-M+1} b_M = 0, \\ c_{L+2} + c_{L+1} b_1 + \cdots + c_{L-M+2} b_M = 0, \\ \vdots \\ c_{L+M} + c_{L+M-1} b_1 + \cdots + c_L b_M = 0, \end{cases} \tag{20}$$

where $c_n = 0$ for $n < 0$ and $b_j = 0$ for $j > M$.

If Eq. (19) and Eq. (20) are non-singular, then we can solve them directly

$$[L/M] = \frac{\begin{vmatrix} c_{L-M+1} & c_{L-M+2} & \cdots & c_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=M}^{L} c_{j-M} x^j & \sum_{j=M-1}^{L} c_{j-M+1} x^j & \cdots & \sum_{j=0}^{L} c_j x^j \end{vmatrix}}{\begin{vmatrix} c_{L-m+1} & c_{L-m+2} & \cdots & c_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{L+1} & c_{L+1} & \cdots & c_{L+M} \\ x^M & x^{M-1} & \cdots & 1 \end{vmatrix}}.$$

If the lower index on a sum exceeds the upper, the sum is replaced by zero. Alternate forms are

$$[L/M] = \sum_{j=0}^{L-M} c_j x^j + x^{L-M+1} w_{L/M}^T W_{L/M}^{-1} w_{L/M} = \sum_{j=0}^{L+n} c_j x^j + x^{L+n+1} w_{(L+M)/M}^T W_{L/M}^{-1} w_{(L+n)/M} \ ,$$

for

$$W_{L/M} = \begin{bmatrix} c_{L-M+1} - xc_{L-M+2} & \cdots & c_L - xc_{L+1} \\ \vdots & \ddots & \vdots \\ c_L - xc_{L+1} & \cdots & c_{L+M+1} - xc_{L+M} \end{bmatrix},$$

$$W_{L/M} = \begin{bmatrix} c_{L-M+1} \\ c_{L-M+2} \\ \vdots \\ c_L \end{bmatrix}.$$

The construction$[L/M]$ of approximants can be made only by algebraic operations [22]. Each choice of L degree of the numerator and M degree of the denominator, leads to an approximant. The significant difficulty in applying the technique is how to direct the choice to obtain the best approximant. This needs the use of a criterion for the choice depending on the shape of the solution. A criterion that has worked well here is the choice of $[L/M]$ approximant.

5. The Hybrid Method Algorithm

Consider a general nonlinear non-homogenous partial differential equation with initial conditions of the form:

$$\mathcal{L}u(x,\mathrm{y},t) + \mathcal{R}u(x,\mathrm{y},t) + \mathcal{N}u(x,\mathrm{y},t) = g(x,\mathrm{y},t), \quad (21)$$

with initial condition

$$u(x,y,0) = h(x,y), \quad (22)$$

where $\mathcal{L} = \frac{\partial}{\partial t}$, $\mathcal{R}$ is a linear differential operator which has partial derivatives, $\mathcal{N}$ is nonlinear operator and $g(x,y,t)$ is source term.

Taking the Yang transform on both sides to Eq. (21), to get:

$$Y[\,\mathcal{L}u(x,\mathrm{y},t)] + Y[\mathcal{R}\,u(x,\mathrm{y},t)] + Y[\mathcal{N}u(x,\mathrm{y},t)] = Y[g(x,\mathrm{y},t)]. \quad (23)$$

Using the differentiation property of the Yang transform to Eq. (23) and above the initial conditions, we have:

$$Y[u(x,\mathrm{y},t)] = sY[g(x,\mathrm{y},t)] + sh(x,y) - sY[\mathcal{R}\,u(x,\mathrm{y},t) + \mathcal{N}u(x,\mathrm{y},t)], \quad (24)$$

applying the inverse Yang transform on both sides to Eq. (24) to find

$$u(x,\mathrm{y},t) = G(x,\mathrm{y},t) - Y^{-1}\{sY[\mathcal{R}u(x,y,t) + \mathcal{N}u(x,y,t)]\}, \quad (25)$$

where $G(x,\mathrm{y},t)$ represents the term arising from the source term and the prescribed initial conditions. Now, we apply the reduced differential transform method:

$$U(x,y,0) = G(x,\mathrm{y},t), \quad (26)$$

$$U(x,y,k+1) = -Y^{-1}\{sY[\mathcal{R}U(x,y,k) + \mathcal{N}U(x,y,k)]\}, \quad (27)$$

where $\mathcal{R}U(x,y,k), \mathcal{N}U(x,y,k)$ are the transformations of the functions $\mathcal{R}u(x,y,t), \mathcal{N}u(x,y,t)$, respectively.

This is coupling of the Yang transform and the reduced differential transform method, then by YRDTM, we have the solution of Eq. (21), with initial condition (22) in the form of infinite series as follows:

$$u(x,y,t) = \sum_{k=0}^{\infty} U(x,\text{y},k).$$

After that, we applied its Padé approximant of an order $[L/M]$ on the power series solution by following the same steps mentioned earlier in Eq. (4). The values L and M are arbitrarily selected. At this stage, the Padé approximant improves the accuracy, and convergence of the truncated series solution by expanding the domain of that solution.

6. Application

In this part, we will illustrate the efficiency and accuracy of the PYRDTM discussed in previous sections using three test problems for the unsteady state two-dimensional convection-diffusion equation.

Problem I (The unsteady state two-dimensional convection - diffusion equation) [28]

Consider Eq. (1) with $\beta_x = \beta_y = -1$, $and\ L = 1$.Then Eq. (1) can be written as:

$$\frac{\partial u}{\partial t}(x,y,t) - \frac{\partial u}{\partial x}(x,y,t) - \frac{\partial u}{\partial y}(x,y,t) - \alpha_x \frac{\partial^2 u}{\partial x^2}(x,y,t) - \alpha_y \frac{\partial^2 u}{\partial y^2}(x,y,t) = 0, \tag{28}$$

with initial condition $u(x,y,0) = a(e^{-c_x \cdot x} + e^{-c_y \cdot y}), 0 \leq x, y \leq 1, t > 0,$ (29)

where, $c_x = \frac{1 \pm \sqrt{1+4b\alpha_x}}{2\alpha_x} > 0, c_y = \frac{1 \pm \sqrt{1+4b\alpha_y}}{2\alpha_y} > 0.$

By taking the Yang transform on both sides to Eq. (28) subject to initial condition (29), we have

$$Y[u(x,y,t)] = Su(0) + SY\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \alpha_x \frac{\partial^2 u}{\partial x^2} + \alpha_y \frac{\partial^2 u}{\partial y^2}\right], \tag{30}$$

by applying the inverse Yang transform to Eq. (30), we have

$$u(x,y,t) = u(0) + Y^{-1}\left(SY\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \alpha_x \frac{\partial^2 u}{\partial x^2} + \alpha_y \frac{\partial^2 u}{\partial y^2}\right]\right), \tag{31}$$

applying the reduced differential transform method to Eq. (31), we have

$$U(x,y,k+1) = Y^{-1}\left(S\,Y\left[\frac{\partial}{\partial x}U(x,y,k) + \frac{\partial}{\partial y}U(x,y,k) + \alpha_x \frac{\partial}{\partial x^2}U(x,y,k) + \alpha_y \frac{\partial}{\partial y^2}U(x,y,k)\right]\right), \tag{32}$$

with $U(x,y,0) = ae^{bt}(e^{-c_x x} + e^{-c_y y}).$ (33)

From relationships (32) and (33), we obtain

$$\begin{aligned} U(x,y,1) &= Y^{-1}\left(S\,Y\left[\frac{\partial}{\partial x}U(x,y,0)+\frac{\partial}{\partial y}U(x,y,0)+\alpha_x\frac{\partial}{\partial x^2}U(x,y,0)+\alpha_y\frac{\partial}{\partial y^2}U(x,y,0)\right]\right), \\ &= a\,t\left(c_x\,\beta\,e^{-c_x\cdot x}+\ c_y\,\xi\,e^{-c_y\cdot y}\right), \end{aligned} \tag{34}$$

$$\begin{aligned} U(x,y,2) &= Y^{-1}\left(S\,Y\left[\frac{\partial}{\partial x}U(x,y,1)+\frac{\partial}{\partial y}U(x,y,1)+\alpha_x\frac{\partial}{\partial x^2}U(x,y,1)+\alpha_y\frac{\partial}{\partial y^2}U(x,y,1)\right]\right), \\ &= \frac{t^2}{2}a(c_x^2\,\beta^2e^{-c_x\cdot x}+c_y^2\,\xi^2e^{-c_y\cdot y}), \end{aligned} \tag{35}$$

$$\begin{aligned} U(x,y,3) &= Y^{-1}\left(S\,Y\left[\frac{\partial}{\partial x}U(x,y,2)+\frac{\partial}{\partial y}U(x,y,2)+\alpha_x\frac{\partial}{\partial x^2}U(x,y,2)+\alpha_y\frac{\partial}{\partial y^2}U(x,y,2)\right]\right), \\ &= \frac{t^3}{6}a(c_x^3\,\beta^3e^{-c_x\cdot x}+c_y^3\,\xi^3e^{-c_y\cdot y})\ , \end{aligned} \tag{36}$$

the solutions series obtained by YRDTM is

$$\begin{aligned} u(x,y,t) &= \textstyle\sum_{k=0}^{\infty}U\,(x,y,k) \\ &= a(e^{c_x.x}+e^{-c_y.y})+a\,t\left(c_x\,\beta\,e^{c_xx}+c_y\,\xi\,e^{-c_y,y}\right)+a\frac{t^2}{2}(c_x^2\beta^2e^{-c_xx}+\ c_y^2\,\xi^2e^{-c_y,y})+.\cdots \end{aligned} \tag{37}$$

The $[L/M]t$-Padé approximant of (37), with $L=1, M=2$

$$\begin{aligned} \left[\frac{L}{M}\right]_u(x,y,t) = \Big(2\mathrm{a}\Big(\Big(\big(&-4\alpha_x^3c_x^6+12\alpha_x^2c_x^5+\left(12\alpha_x^2\,\alpha_yc_y^2-12\alpha_x^2c_y-12\alpha_x\right)c_x^4+\ \left(-24\alpha_x\alpha_yc_y^2+\right. \\ &\left.24\alpha_xc_y+4\right)c_x^3-6c_y\xi\left(\alpha_x\alpha_yc_y^2-\alpha_xc_y-2\right)c_x^2\ +6c_y^2c_x\xi^2+\ c_y^3\xi^3\big)t+ \\ &12c_y\xi\left(\alpha_xc_x^2-\frac{1}{4}\alpha_yc_y^2-c_x+\frac{1}{4}c_y\right)\Big)e^{-2c_xx}+\Big(\big(\alpha_x^3c_x^6-3\alpha_x^2c_x^5+\ \left(-6\alpha_x^2\alpha_yc_y^2\ +\right. \\ &\left.6\alpha_x^2c_y+3\alpha_x\right)c_x^4+\left(12\alpha_x\alpha_yc_y^2-12\alpha_xc_y-1\right)c_x^3+12c_y\left(c_y^2\alpha_y\ -\alpha_xc_y\frac{1}{2}\right)\xi\,c_x^2- \\ &12c_y^2c_x\xi^2c_x-4c_y^3\xi^3\big)\,t-3c_x\beta\left(\alpha_xc_x^2-4\alpha_yc_y^2-c_x+4c_y\right)\Big)e^{-c_xx}e^{-2c_y,y}+ \\ &\beta^2c_x^4(3+(\alpha_xc_x^2-c_x)t)(e^{-3c_x.x}+e^{-c_y.y})+\ c_y^2\xi^2\left(3+\left(\alpha_yc_y^2\ -c_y\right)t\right)\Big)\Big)\Big/ \\ &\Big(\Big(-2(c_x^4\alpha_x^2-2\alpha_xc_x^3+\Big(\left(-3\alpha_x\alpha_yc_y^2+3\alpha_xc_y+1\right)c_x^2+\left(3\alpha_yc_y^2-3c_y\right)c_x+ \\ &\xi^2\Big)\xi^2t^2\ +2\left(\alpha_xc_x^2+\alpha_yc_y^2-c_x-c_y\right)\left(\alpha_x^2c_x^4-2\alpha_xc_x^3+\left(-4\alpha_x\alpha_yc_y^2+\right.\right. \\ &\left.\left.4\alpha_xc_y+1\right)c_x^2+\left(4\alpha_yc_y^2-4c_y\right)c_x+c_y^2\right)t-6\alpha_x^2c_x^4+\ 12\alpha_xc_x^2\ +\left(24\alpha_xc_yc_y^2-\right. \\ &\left.24\alpha_xc_y-6\right)c_x^2+\left(-24\alpha_yc_y^2+24c_y\right)c_x-\ (\ 6c_y^2\xi^2\ )\,e^{-c_y\cdot y-c_x\cdot x}+c_x^2(6+c_x^2\beta^2t^2+ \\ &\beta^2(-4\alpha_xc_x^2\ +4c_x)t)e^{-2c_x\cdot x}+\ c_y^2\left(6+\ c_y^2\xi^2t^2+\ \xi^2\left(-4\alpha_yc_y^2+4c_y\right)t\right)e^{-2c_y\cdot y})\Big), \end{aligned} \tag{38}$$

where ξ= (α_yc_y-1), and $\beta=(\alpha_xc_x-1)$.

Problem (II) (System of two-dimensional Burgers' equations) [29]

Consider the Eq. (4) and Eq. (5), with $\alpha=1$. Then these equations become

$$\begin{aligned} \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} &= \frac{1}{Re}\left(\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}\right), \\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} &= \frac{1}{Re}\left(\frac{\partial^2v}{\partial x^2}+\frac{\partial^2v}{\partial y^2}\right), \end{aligned} \tag{39}$$

with initial conditions

PII-1: $u(x,y,0) = \frac{3}{4} - \frac{1}{4[1+\exp(\omega(x,y))]}, v(x,y,0) = \frac{3}{4} + \frac{1}{4[1+\exp(\omega(x,y))]}$, (40)

by taking the Yang transform on both sides to Eq. (39) subject to initial condition (40), we have

$$Y[u(x,y,t)] = Su(0) + SY\left[-u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right], \quad (41)$$

$$Y[u(x,y,t)] = Su(0) + SY\left[-u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right], \quad (42)$$

by applying the inverse Yang transform to Eq. (41) and Eq. (42), we have

$$u(x,y,t) = u(0) + Y^{-1}\left(S\,Y\left[-u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right]\right), \quad (43)$$

$$v(x,y,t) = v(0) + Y^{-1}\left(S\,Y\left[-u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\right]\right), \quad (44)$$

applying the reduced differential transform method to Eq. (43) and Eq. (44), we obtain

$$U(x,y,k+1) = Y^{-1}\left(S\,Y\left[-\sum_{r=0}^{k} U(x,y,r)\frac{\partial}{\partial x}U(x,y,k-r) - \sum_{r=0}^{k} V(x,y,r)\frac{\partial}{\partial y}U(x,y,k-r) + \frac{1}{Re}\left(\frac{\partial^2}{\partial x^2}U(x,y,k) + \frac{\partial^2}{\partial y^2}U(x,y,k)\right)\right]\right), \quad (45)$$

$$V(x,y,k+1) = Y^{-1}\left(S\,Y\left[-\sum_{r=0}^{k} U(x,y,r)\frac{\partial}{\partial x}V(x,y,k-r) - \sum_{r=0}^{k} V(x,y,r)\frac{\partial}{\partial y}V(x,y,k-r) + \frac{1}{Re}\left(\frac{\partial^2}{\partial x^2}V(x,y,k) + \frac{\partial^2}{\partial y^2}V(x,y,k)\right)\right]\right), \quad (46)$$

with

$$U(x,y,0) = u(x,y,0) = \frac{3}{4} - \frac{1}{4\omega^+}, \quad (47)$$

$$V(x,y,0) = v(x,y,0) = \frac{3}{4} + \frac{1}{4\omega^+}, \quad (48)$$

from relationships (45), (46), (47), and (48) give us the values of $U(x,y,k)$ and $V(x,y,k)$ as follows:

$$U(x,y,0) = \frac{3}{4} - \frac{1}{4\omega^+}, \quad (49)$$

$$U(x,y,1) = -Re\, e^{\omega(x,y)}/128(\omega^+)^2 t\,, \quad (50)$$

$$U(x,y,2) = -Re^2 e^{\omega(x,y)}(-\omega^-)/8192(\omega^+)^3 t^2, \quad (51)$$

and

$$V(x,y,0)=\frac{3}{4}+\frac{1}{4\omega^{+}}, \tag{52}$$

$$V(x,y,1)=\ Re\ e^{\omega(x,y)}/128(\omega^{+})^{2}t, \tag{53}$$

$$V(x,y,2)=Re^{2}e^{\omega(x,y)}(-\omega^{-})/8192(\omega^{+})^{3}\ t^{2}, \tag{54}$$

the solution obtained by YRDTM is

$$\begin{aligned} u(x,y,t) &= \sum_{k=0}^{\infty}U(x,y,k)=\frac{3}{4}-\frac{1}{4\omega^{+}}+\frac{-Re^{\omega(x,y)}}{128(\omega^{+})^{2}}t-\frac{Re^{2}e^{\omega(x,y)}(-\omega^{-})}{8192(\omega^{+})^{3}}t^{2}+\ldots \\ &= \frac{3}{4}-1/4[1+\exp(\omega(x,y)-(Re/32)t)]. \end{aligned} \tag{55}$$

$$\begin{aligned} v(x,y,t) &= \sum_{k=0}^{\infty}V(x,y,k)=\frac{3}{4}+\frac{1}{4\omega^{+}}+\frac{Re\ e^{\omega(x,y)}}{128(\omega^{+})^{2}}t+\frac{Re^{2}e^{\omega(x,y)}(-\omega^{-})}{8192(\omega^{+})^{3}}t^{2}+\ldots \\ &= \frac{3}{4}+1/4[1+\exp(\omega(x,y)-(Re/32)t)]. \end{aligned} \tag{56}$$

All of the $[L/M]t$-Padé approximant of (55) and (56), with $L=3, M=2$ yield

$$\left[\frac{L}{M}\right]_{u}(x,y,t)= \frac{(288Re^{2}t^{2}-73728Ret+5898240)e^{2\omega(x,y)}+(-Re^{3}t^{3}+1440Re^{2}t^{2}-36864Ret+9830400)\ e^{\omega(x,y)}+192Re^{2}t^{2}+49152Re\ t+3932160}{((384Re^{2}t^{2}-98304Ret+7864320)e^{2\omega(x,y)}+(2304Re^{2}t^{2}+15728640)e^{\omega(x,y)}+384Re^{2}t^{2}+98304Ret+7864320},$$

$$\left[\frac{L}{M}\right]_{v}(x,y,t)= \frac{(288Re^{2}t^{2}-73728Ret+5898240)e^{2\omega(x,y)}+(Re^{3}t^{3}+2016Re^{2}t^{2}+36864Ret+13762560)\ e^{\omega(x,y)}+384Re^{2}t^{2}+98304Ret+7864320}{((384Re^{2}t^{2}-98304Ret+7864320)e^{2\omega(x,y)}+(2304Re^{2}t^{2}+15728640)e^{\omega(x,y)}+384Re^{2}t^{2}+98304Re\ t+7864320},$$

where $\omega(x,y)=\frac{Re(y-x)}{8}$, $\omega^{-}=1-\exp\,(\omega(x,y))$ and $\omega^{+}=1+\exp\,(\omega(x,y))$.

Problem (II) (System of two-dimensional Burgers' equations)

Consider the same system that is given in Eq. (39) with initial conditions:

PII-2: $u(x,y,0)=S^{x}S^{y}, v(x,y,0)=(S^{x}+2S^{x}C^{x})(S^{y}+2S^{y}C^{y})$, (57)

Where $S^{x}=\sin(\pi x)\,, S^{y}=\sin(\pi y)\,, C^{x}=\cos\,(\pi x),\ C^{y}=\cos\,(\pi y)$. (58)

By taking the Yang transform on both sides to Eq. (39) subject to the initial condition (57), we have

$$Y[u(x,y,t)]=Su(0)+SY\left[-u\frac{\partial u}{\partial x}-v\frac{\partial u}{\partial y}+\frac{1}{Re}\left(\frac{\partial^{2}u}{\partial x^{2}}+\frac{\partial^{2}u}{\partial y^{2}}\right)\right], \tag{59}$$

$$Y[v(x,y,t)]=Sv(0)+SY\left[-u\frac{\partial v}{\partial x}-v\frac{\partial v}{\partial y}+\frac{1}{Re}\left(\frac{\partial^{2}v}{\partial x^{2}}+\frac{\partial^{2}v}{\partial y^{2}}\right)\right], \tag{60}$$

by applying the inverse Yang transform to Eq. (59) and Eq. (60), we have

$$u(x,y,t) = u(0) + Y^{-1}\left(S\,Y\left[-u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right]\right), \tag{61}$$

$$v(x,y,t) = v(0) + Y^{-1}\left(S\,Y\left[-u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\right]\right), \tag{62}$$

applying the reduced differential transform method to Eq. (61) and Eq. (62), we have

$$U(x,y,k+1) = Y^{-1}\left(S\,Y\left[-\sum_{r=0}^{k} U(x,y,r)\frac{\partial}{\partial x}U(x,y,k-r) - \sum_{r=0}^{k} V(x,y,r)\frac{\partial}{\partial y}U(x,y,k-r) + \frac{1}{Re}\left(\frac{\partial^2}{\partial x^2}U(x,y,k) + \frac{\partial^2}{\partial y^2}U(x,y,k)\right)\right]\right), \tag{63}$$

$$V(x,y,k+1) = Y^{-1}\left(S\,Y\left[-\sum_{r=0}^{k} U(x,y,r)\frac{\partial}{\partial x}V(x,y,k-r) - \sum_{r=0}^{k} V(x,y,r)\frac{\partial}{\partial y}V(x,y,k-r) + \frac{1}{Re}\left(\frac{\partial^2}{\partial x^2}V(x,y,k) + \frac{\partial^2}{\partial y^2}V(x,y,k)\right)\right]\right), \tag{64}$$

with

$$U(x,y,0) = S^x S^y, \tag{65}$$

$$V(x,y,0) = (S^x + 2S^x C^x)(S^y + 2S^y C^y), \tag{66}$$

from relationships (63), (64), (65) and (66), give us the values of $U(x,y,k)$ and $V(x,y,k)$ as follows:

$$U(x,y,0) = S^x S^y, \tag{67}$$

$$U(x,y,1) = -t\,\pi S^x[2ReC^y S^x C^x(S^y + 2S^y C^y) + 2ReS^x S^y (C^y)^2 - ReS^y (C^y)^2 + ReS^x S^y C^y + ReC^y + 2\pi S^y]/Re \tag{68}$$

and

$$V(x,y,0) = (S^x + 2S^x C^x)(S^y + 2S^y C^y), \tag{69}$$

$$\begin{aligned} V(x,y,0) = -t/Re\Big[4\pi\Big(\Big(&Re\left(2(S^x)^2C^x + 1 - \left(\tfrac{1}{4}\right)(C^x)^2((C^x)^2 - (S^x)^2)^2\right)((C^y)^2 - (S^y)^2) \\ &- \left(\tfrac{1}{4}\right)ReC^y((C^x)^2 - (S^x)^2)^2 + \left(\tfrac{1}{2}\right)ReS^xS^y((C^x)^2 - (S^x)^2) + \left(\left(\tfrac{1}{2}\right)ReS^xC^y + \right. \\ &\left.2\pi\right)2S^xC^x + \left(\left(\left(\tfrac{1}{4}\right)ReS^yC^x + \left(\tfrac{5}{4}\right)\pi\right)C^x - \left(\tfrac{1}{4}\right)ReC^y(C^x)^2 - 2)\right)2S^yC^y \\ &+ReS^y\left(2(S^x)^2C^x + 1 - \left(\tfrac{1}{2}\right)(C^x)^2 - \left(\tfrac{1}{2}\right)((C^x)^2 - (S^x)^2)^2\right)((C^y)^2 - (S^y)^2 - \\ &\left(\tfrac{1}{4}\right)ReS^yC^y((C^x)^2 - (S^x)^2 - \left(\tfrac{1}{2}\right)ReS^x(C^y - 1)(C^y + 1)((C^x)^2 - (S^x)^2) \\ &+2\left(\tfrac{1}{2}\left(ReS^xC^y + \left(\tfrac{5}{2}\right)\pi\right)\right)(S^x)^2C^x + \left(\left(\tfrac{1}{2}\right)\pi S^y - \left(\tfrac{1}{4}\right)ReC^x(C^y - 1)(C^y + 1)\right)S^x \\ &- \left(\tfrac{1}{4}\right)ReS^yC^y((C^x)^2 - 2)\Big)\Big]. \end{aligned} \tag{70}$$

The solution obtained by YRDTM is

$$u_1(x,y,t) = \sum_{k=0}^{1} U(x,y,k). \tag{71}$$

$$v_1(x,y,t) = \sum_{k=0}^{1} V(x,y,k). \tag{72}$$

All of the $[L/M]t$-Padé approximant of (71) and (72), with $L = 0, M = 1$

$$\left[\frac{L}{M}\right] u(x,y,t) = (ReS^x(S^y)^2)/(ReS^y + \pi Re(C^yS^xS^y) + 2\pi Re\, S^xS^y(C^y)^2 + 2\pi Re\, S^yS^xC^xC^y - \pi Re\, C^x(C^y)^2 + 4\pi Re\, S^xC^xS^y(C^y)^2 + 2\pi^2 S^y + Re\, \pi C^x)t), \tag{73}$$

$$\left[\frac{L}{M}\right] v(x,y,t) = ((S^x)^2(S^y)^2Re + 4Re(S^x)^2C^y(S^y)^2 + 4Re(S^x)^2(S^y)^2(C^y)^2 + 4Re(S^y)^2(S^x)^2C^x + 16Re\,(S^x)^2(S^y)^2C^xC^y + 16Re\,C^x(S^x)^2(C^y)^2(S^y)^2 + 2Re\,(S^y)^2S^xC^x + 16Re\,(S^x)^2(S^y)^2C^y(C^x)^2 + 16Re\,(S^x)^2(S^y)^2(C^y)^2(C^x)^2) /(Re\,S^xS^y + 2Re\,S^yS^xC^y + 2Re\,S^yS^xC^x + 4Re\,S^xC^xS^yC^y + 2\pi Re\,((C^x)^2 - (S^x)^2)S^x\pi + 4\pi Re\,((C^x)^2 - (S^x)^2)S^y + 8\pi Re((C^y)^2 - (S^y)^2))S^yC^y + \pi Re\,S^xC^x + 2\pi ReS^yC^y + 2\pi Re\,S^y(C^y)^2 - 8\pi ReS^y(S^x)^2(C^x)^2((C^y)^2 - (S^y)^2) - 2\pi Re\,S^yC^y((C^x)^2 - (S^x)^2)^2((C^y)^2 - (S^y)^2) - \pi Re\,S^yC^y((C^x)^2 - (S^x)^2)^2 - 4\pi Re((C^x)^2 - (S^x)^2)^2S^y(C^y)^2 - 2\pi Re\,C^x(S^x)^2(C^y)^2 - 2\pi Re((C^y)^2 - (S^y)^2)S^y((C^x)^2 - (S^x)^2) - 4\pi Re\,(C^x)^2S^yC^y((C^y)^2 - (S^y)^2)(C^x)^2 - \pi ReS^xC^x(C^y)^2 - 2\pi S^yRe(C^x)^2(C^y)^2 + 2\pi^2S^xS^y + 10\pi^2S^xS^yC^y + 10\pi^2S^yS^xC^x + 32\pi^2S^yS^xC^xS^yC^y + 4\pi ReS^xC^y(S^y)^2((C^x)^2 - (S^x)^2) + 8\pi Re\,S^yC^x((C^y)^2 - (S^y)^2)(S^x)^2 + 8\pi Re\,((C^y)^2 - 4Re\,S^yC^yC^x(S^y)^2)(S^x)^2C^x + 2\pi ReS^xC^xC^y(S^y)^2 + 2\pi Re\,C^x(S^x)^2(S^y)^2 + 8\pi Re\,C^xS^y(S^x)^2(C^y)^2)t). \tag{74}$$

7. Results and Discussions

The proposed PYRDTM is a powerful new method to find the analytical solutions for three test problems for the unsteady state two-dimensional convection-diffusion equations. All calculations are run by Maple 2016 software.

Figure 1 to Figure 3 show the exact solution, PYRDTM solution and absolute errors at $(t = 0.1, \alpha_x = \alpha_y = 0.1)$and $(t = 0.5, R = 100)$ respectively for problems **(I)** and **(II)**; it should be note that all figures calculated at Padé $[1/2]$ for problem **(I)** at three iteration and at Padé $[3/2]$ for problem **(II)** at five iterations.

Figure 4 show the PYRDTM $[0/1]$ of u and PYRDTM $[0/2]$ of v at $(t = 0.01, R = 1)$ for problems **(II)** with second initial condition **PII-2**.

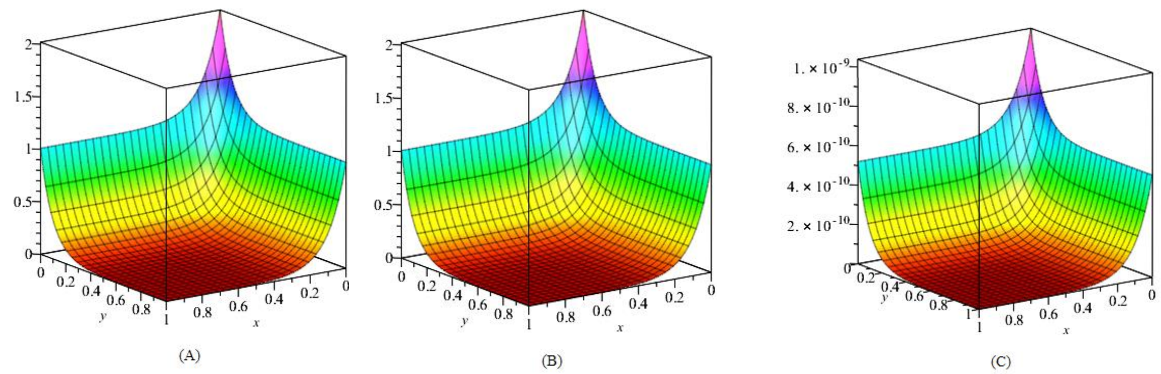

Fig. 1. (A) Exact solution, (B) PYRDTM solution $[1/2]$, (C) Absolute error at $t = 0.1$ and $\alpha_x = \alpha_y = 0.1$, Problem **(I)**

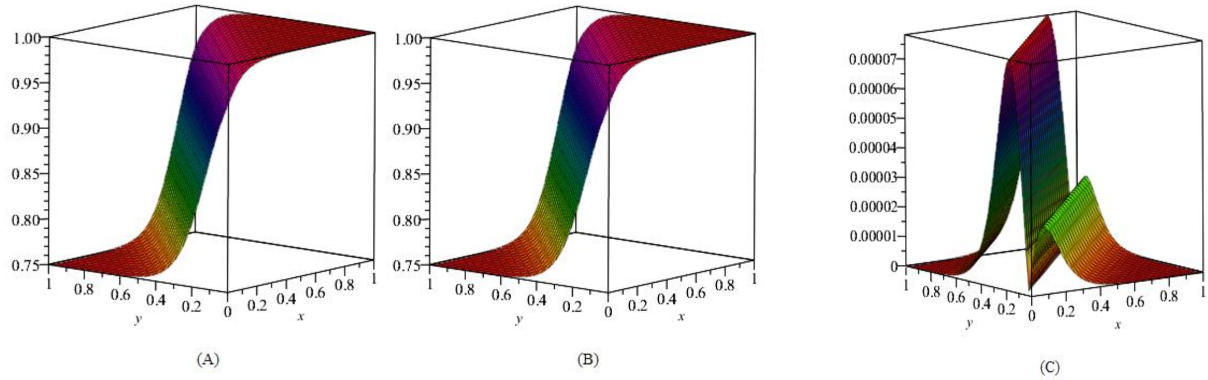

Fig. 2. (A) Exact solution (B) PYRDTM $[3/2]$ solution (C) Absolute error at $t = 0.5$ and $R = 100$, problem **(II)**

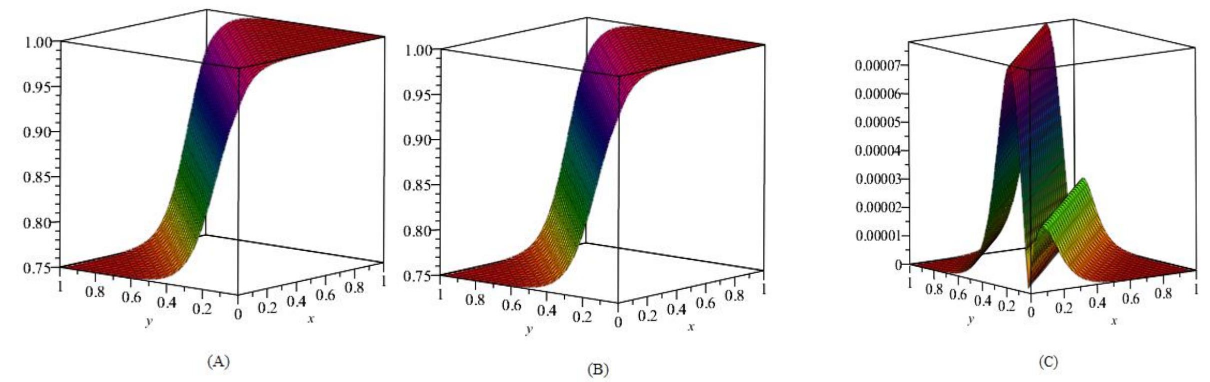

Fig. 3. (A) Exact solution (B) PYRDTM $[3/2]$ solution (C) Absolute error at $t = 0.5$ and $R = 100$, problem **(II)**

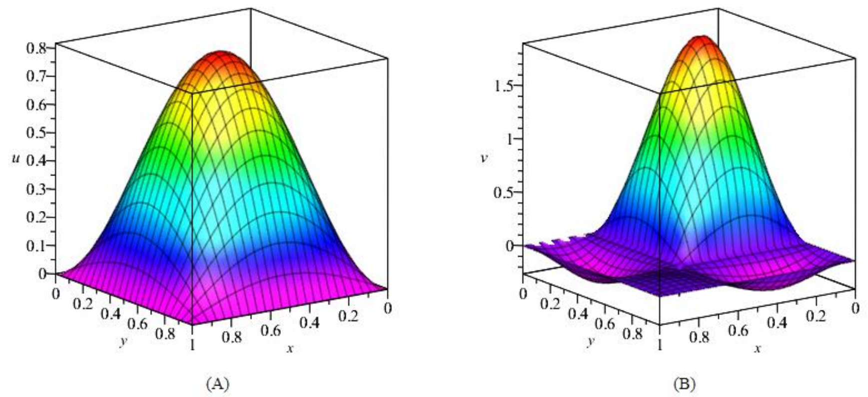

Fig. 4. (A) PYRDTM $[0/1]$ solution of u , (B) PYRDTM $[0/2]$ solution of v at $t = 0.01$ and $R = 1$, Problem **(II)**

Table 1 show the comparison of numerical results in terms of the largest errors and CPU time between PYRDTM $[1/2]$, RDTM, MDQM [3] and LDQM [3] at $(t = 0.1, \alpha_x = \alpha_y = 0.1)$ for different values of h. The results confirm that the three-iteration PYRDTM is more accurate and has a lower CPU time than other methods.

Table 1
Comparison of errors and CPU time between LDQM, BDQM, RDTM and PYRDTM $[1/2]$ at $t = 0.1$ and $\alpha_x = \alpha_y = 0.1$, Problem **(I)**

h	LDQM [3]		BDQM [3]		PYRDTM		RDTM	
	L∞	CPU	L∞	CPU	L∞	CPU	L∞	CPU
0.25	2.57E − 06	0.509	4.16E − 07	0.509	1.04E − 09	0.016	1.03E − 02	0.031
0.17	1.38E − 05	0.619	5.64E − 06	0.619	1.04E − 09	0.016	1.05E − 02	0.031
0.125	3.22E − 05	0.772	7.95E − 06	0.772	1.04E − 09	0.016	1.08E − 02	0.031
0.1	5.35E − 05	1.047	9.18E − 06	1.047	1.04E − 09	0.031	1.13E − 02	0.047

Table 2 to Table 5 show the comparison of absolute errors between PYRDTM, RDTM, Bahadır [6], ELMFS [30] and DADM [31]. Therefore, the absolute error results confirm that the four-iteration PYRDTM is performs better than RDTM, Bahadır [6], ELMFS [30], and DADM [31].

Table 2
Comparison of absolute errors among different methods and PYRDTM $[3/2]$ for $u(x, y, t)$ at $R = 100\ and\ t = 0.01$, Problem **(II)**

(x, y)	PYRDTM	RDTM	ELMFS [30]	DADM [31]	Bahadır [6]
(0.1 ,0.1)	$1.80E-17$	$1.53E-15$	$7.58E-04$	$5.91E-05$	$7.24E-05$
(0.5 ,0.1)	$2.07E-16$	$1.31E-15$	$8.37E-05$	$4.84E-06$	$2.43E-05$
(0.9 ,0.1)	$1.43E-18$	$1.46E-17$	$6.01E-05$	$3.41E-08$	$8.39E-06$
(0.3,0.3)	$1.80E-17$	$1.53E-15$	$2.98E-05$	$5.91E-05$	$8.25E-05$
(0.7 ,0.3)	$2.07E-16$	$1.31E-15$	$2.17E-05$	$4.84E-06$	$3.43E-05$
(0.1 ,0.5)	$2.18E-16$	$1.31E-15$	$1.27E-05$	$1.64E-06$	$5.62E-05$
(0.5 ,0.5)	$1.80E-17$	$1.53E-15$	$2.68E-05$	$5.91E-05$	$7.33E-05$
(0.9,0.5)	$2.07E-16$	$1.31E-15$	$3.68E-05$	----	----
(0.3,0.7)	$2.18E-16$	$1.31E-15$	$7.19E-05$	----	----
(0.7,0.7)	$1.80E-17$	$1.53E-15$	$3.53E-05$	----	----
(0.1,0.9)	$1.51E-18$	$1.47E-17$	$6.74E-05$	----	----
(0.5,0.9)	$2.18E-16$	$1.31E-15$	$1.29E-05$	----	----
(0.9,0.9)	$1.80E-17$	$1.53E-15$	$3.62E-04$	----	----

Table 3
Comparison of absolute errors among different methods and PYRDTM $[3/2]$ for $u(x,y,t)$ at $R = 100\ and\ t = 0.5$, Proplem **(II)**

(x,y)	PYRDTM	RDTM	ELMFS [30]	DADM [31]	Bahadır [6]
(0.1 ,0.1)	$8.87E-06$	$9.61E-04$	$9.12E-04$	$2.78E-4$	$5.13E-4$
(0.5 ,0.1)	$1.02E-06$	$1.83E-05$	$1.17E-04$	$4.52E-4$	$8.86E-4$
(0.9 ,0.1)	$6.97E-09$	$1.86E-07$	$7.65E-06$	$3.37E-6$	$6.53E-5$
(0.3,0.3)	$8.87E-06$	$9.61E-04$	$9.48E-04$	$2.78E-4$	$7.32E-4$
(0.7 ,0.3)	$1.02E-06$	$1.83E-05$	$5.40E-05$	$4.52E-4$	$6.27E-4$
(0.1 ,0.5)	$1.29E-05$	$2.09E-05$	$4.84E-06$	$2.87E-4$	$4.02E-4$
(0.5 ,0.5)	$8.87E-06$	$9.61E-04$	$3.39E-04$	$2.78E-4$	$3.47E-4$
(0.9,0.5)	$1.02E-06$	$1.83E-05$	$5.60E-05$	----	-----
(0.3,0.7)	$1.29E-05$	$2.09E-05$	$1.51E-05$	----	----
(0.7,0.7)	$8.87E-06$	$9.61E-04$	$3.45E-04$	-----	----
(0.1,0.9)	$9.36E-08$	$2.92E-07$	$3.92E-05$	----	----
(0.5,0.9)	$1.29E-05$	$2.09E-05$	$8.93E-05$	----	----
(0.9,0.9)	$8.87E-06$	$9.61E-04$	$4.44E-04$	----	----

Table 4
Comparison of absolute errors among different methods and PYRDTM $[3/2]$ for $v(x,y,t)$ at $R = 100\ and\ t = 0.01$, Problem **(II)**

(x,y)	PYRDTM	RDTM	ELMFS [30]	DADM [31]	Bahadır [6]
(0.1 ,0.1)	$1.80E-17$	$1.53E-15$	$5.51E-04$	$5.91E-05$	$8.36E-05$
(0.5 ,0.1)	$2.07E-16$	$1.31E-15$	$1.84E-05$	$4.84E-06$	$5.14E-05$
(0.9 ,0.1)	$1.43E-18$	$1.46E-17$	$1.06E-04$	$3.41E-08$	$7.03E-06$
(0.3,0.3)	$1.80E-17$	$1.53E-15$	$1.65E-05$	$5.91E-05$	$6.15E-05$
(0.7 ,0.3)	$2.07E-16$	$1.31E-15$	$5.37E-06$	$4.84E-06$	$5.41E-05$
(0.1 ,0.5)	$2.18E-16$	$1.31E-15$	$8.13E-05$	$1.64E-06$	$7.35E-05$
(0.5 ,0.5)	$1.80E-17$	$1.53E-15$	$2.92E-05$	$5.91E-05$	$8.51E-05$
(0.9,0.5)	$2.07E-16$	$1.31E-15$	$1.46E-05$	----	----
(0.3,0.7)	$2.18E-16$	$1.31E-15$	$2.46E-05$	-----	----
(0.7,0.7)	$1.80E-17$	$1.53E-15$	$1.52E-05$	----	----
(0.1,0.9)	$1.51E-18$	$1.47E-17$	$7.84E-05$	----	----
(0.5,0.9)	$2.18E-16$	$1.31E-15$	$2.97E-05$	----	----
(0.9,0.9)	$1.80E-17$	$1.53E-15$	$2.74E-04$	----	----

Table 5
Comparison of absolute errors among different methods and PYRDTM $[3/2]$ for $v(x,y,t)$ at $R = 100\ and\ t = 0.5$, Problem **(II)**

(x,y)	PYRDTM	RDTM	ELMFS [30]	DADM [31]	Bahadır [6]
(0.1 ,0.1)	$8.87E-06$	$9.61E-04$	$4.93E-04$	$2.78E-04$	$6.17E-04$
(0.5 ,0.1)	$1.02E-06$	$1.83E-05$	$1.24E-04$	$4.52E-04$	$4.67E-04$
(0.9 ,0.1)	$6.97E-09$	$1.86E-07$	$2.60E-04$	$3.37E-06$	$1.70E-05$
(0.3,0.3)	$8.87E-06$	$9.61E-04$	$1.16E-03$	$2.78E-04$	$6.25E-04$
(0.7 ,0.3)	$1.02E-06$	$1.83E-05$	$1.23E-04$	$4.52E-04$	$4.66E-04$
(0.1 ,0.5)	$1.29E-05$	$2.09E-05$	$3.01E-04$	$2.87E-04$	$8.72E-04$
(0.5 ,0.5)	$8.87E-06$	$9.61E-04$	$4.73E-04$	$2.78E-04$	$6.23E-04$
(0.9,0.5)	$1.02E-06$	$1.83E-05$	$8.09E-05$	----	----
(0.3,0.7)	$1.29E-05$	$2.09E-05$	$1.86E-04$	----	----
(0.7,0.7)	$8.87E-06$	$9.61E-04$	$3.58E-04$	----	----
(0.1,0.9)	$9.36E-08$	$2.92E-07$	$2.15E-04$	----	----
(0.5,0.9)	$1.29E-05$	$2.09E-05$	$9.33E-05$	----	----
(0.9,0.9)	$8.87E-06$	$9.61E-04$	$3.79E-04$	----	----

Table 6 shows a comparison between PYRDTM $[3/2]$ and RDTM with ADM [32] for five iterations. Therefore, the absolute error results confirm that the five-iteration PYRDTM is performs better than RDTM and ADM.

Table 7 and Table 8 show L_2 and L_∞ of u and v respectively, of problem **(II)** at Padé $[3/2]$ and $(R = 100, t = 1)$. The result confirms that the PYRDTM more accurate compared to other methods.

Table 6
Comparison of absolute errors between PYRDTM $[3/2]$, RDTM and ADM at $R = 100$ and $t = 0.5$, Problem **(II)**

(x, y)	$5 - th\,u$			$5 - th\,v$		
	PYRDTM	RDTM	ADM [32]	PYRDTM	RDTM	ADM [32]
(0.1,1)	$2.68E-08$	$8.38E-08$	$1.82E-05$	$2.68E-08$	$8.38E-08$	$1.82E-05$
(0.2,1)	$9.36E-08$	$2.92E-07$	$6.36E-05$	$9.36E-08$	$2.92E-07$	$6.37E-05$
(0.3,1)	$3.26E-07$	$1.01E-06$	$2.22E-04$	$3.26E-07$	$1.01E-06$	$2.22E-04$
(0.4,1)	$1.13E-06$	$3.40E-06$	$7.72E-04$	$1.13E-06$	$3.40E-06$	$7.75E-04$
(0.5,1)	$3.89E-06$	$1.04E-05$	$2.67E-03$	$3.89E-06$	$1.04E-05$	$1.82E-05$

Table 7
Comparison of errors between PYRDTM $[3/2]$, RDTM and EXP-FDE at $R = 100$ and $t = 1$, Problem **(II),** for u

PYRDTM u			*RDTM* u		*Exp-FDE* u [33]	
$Grid$	$L2$	$L\infty$	$L2$	$L\infty$	$L2$	$L\infty$
4×4	$3.24E-03$	$6.39E-03$	$4.48E-02$	$7.81E-02$	$8.57E-02$	$9.70E-02$
8×8	$2.60E-03$	$6.39E-03$	$3.56E-02$	$7.81E-02$	$4.94E-02$	$4.69E-02$
16×16	$2.50E-03$	$6.39E-03$	$3.68E-02$	$8.98E-02$	$1.92E-02$	$2.05E-02$
32×32	$2.45E-03$	$6.39E-03$	$3.62E-02$	$1.12E-01$	$8.68E-03$	$9.07E-03$

Table 8
Comparison of errors between PYRDTM $[3/2]$, RDTM and EXP-FDE at $R = 100$ and $t = 1$, Problem **(II),** for v

PYRDTM v			*RDTM* v		*Exp-FDE* v [33]	
$Grid$	$L2$	$L\infty$	$L2$	$L\infty$	$L2$	$L\infty$
4×4	$3.24E-03$	$6.39E-03$	$4.48E-02$	$7.81E-02$	$8.57E-02$	$9.70E-02$
8×8	$2.60E-03$	$6.39E-03$	$3.56E-02$	$7.81E-02$	$4.94E-02$	$4.69E-02$
16×16	$2.50E-03$	$6.39E-03$	$3.68E-02$	$8.98E-02$	$1.92E-02$	$2.05E-02$
32×32	$2.45E-03$	$6.39E-03$	$3.62E-02$	$1.12E-01$	$8.69E-03$	$9.08E-03$

Table 9 shows the values of u and v that obtained by using PYRDTM $[1/2]$ are identical with those are given by using RDTM and MQ [32] at least two digits of problem **(II)** with second initial condition **PII-2.** According to the calculations shown in the tables and figures, the PYRDTM procedures are very effective in resolving linear and nonlinear unsteady state two-dimensional convection-diffusion equations; they are also the most efficient; they also provide high-precision solutions since they produce good results with few iterations of solutions and smaller errors with little CPU time.

Table 9
Comparison of the values of u and v between PYRDTM $[1/2]$, RDTM and MQ at $R = 1$ and $t = 0.01$, Problem **(II)**

u				v		
(x, y)	PYRDTM	RDTM	MQ [34]	PYRDTM	RDTM	MQ [34]
(0.1,0.1)	0.71971	0.07346	0.07251	0.43781	0.43295	0.43085
(0.2,0.8)	0.27969	0.27574	0.27778	−0.14893	−0.10644	−0.12409
(0.4,0.4)	0.72242	0.72437	0.72174	1.59935	1.68179	1.65244
(0.7,0.1)	0.20515	0.20741	0.20484	0.56924	0.07022	0.06702
(0.9,0.9)	0.79363	0.79369	0.07945	0.15746	0.01013	0.01334

8. Conclusion

In this work, the PYRDTM has been successfully applied to find analytical solutions for the unsteady state two-dimensional convection-diffusion equation. This technique, which we name PYRDTM, considerably improves the convergence rate of the RDTM truncated series solution. The efficiency and accuracy of the suggested method are illustrated by three test problems. When compared to the most well-known methodologies such as the discrete Adomian decomposition method (DADM) and the Adomian decomposition method (ADM). The results show that PYRDTM is more efficient and accurate. Also, the suggested method's advantage is that it provides a high-accuracy solution with fewer iterations and fewer errors with little CPU time. The suggested method significantly improved the results of the classical RDTM method in error and CPU time. The most important part of this study is that the suggested method is suitable for solving both linear and nonlinear problems.

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