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Biquartic Hesitant Fuzzy Bézier Surface Approximation Model with Its Visualization

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ABSTRACT

Geometric modeling has evolved significantly since its inception, with pioneers like Pierre Bézier laying the foundation for curve modeling. Concurrently, fuzzy set theory, introduced by Lotfi A. Zadeh in 1965, addressed uncertainty in decision-making processes. Integrating fuzzy set theory with geometric modeling has led to advancements in handling imprecise data and uncertainty. This paper proposes a novel approach, the hesitant fuzzy Bézier surface (HFBS) approximation model, which combines geometric modeling with hesitant fuzzy sets to address uncertainty in surface approximation. The model utilizes hesitant fuzzy control net relations to construct HFBSs, enabling visualization of surfaces under varying degrees of uncertainty. A biquartic HFBS example is presented, demonstrating the model's ability to handle hesitancy among experts' opinions. The paper discusses the properties of HFBS and suggests its extension to interpolation models for broader applicability. Ultimately, the HFBS model offers an approach to geometric modeling in situation where uncertainty is inherent, which aim to handle complex data in real-world applications.

1. Introduction

Geometric modeling, pivotal across numerous fields, has significantly influenced modern technological progress since its inception. Pierre Bézier introduced geometric modeling in the 1960s, aiming to mathematically represent intricate shapes and surfaces via parametrization [1]. The history of curve modeling traces back to the seminal work of mathematicians such as Pierre Bézier and Paul de Casteljau in the 1940s and 1950s, who pioneered Bézier curves for automotive design [2]. Simultaneously, mathematicians like Isaac Jacob Schoenberg laid the groundwork for spline functions, which are crucial for curve approximation [3]. The emergence of computational geometry in the 1970s led to advancements by researchers such as Carl de Boor in B-Spline curve techniques [2]. The introduction of NURBS by Dale Myers and John Hart in the 1980s revolutionized curve modeling, providing enhanced flexibility [3]. Geometric modeling finds applications in engineering, computer graphics, and data visualization, facilitating the creation and analysis of complex structures

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across various fields [4,5]. Geometric modeling goes beyond curves to represent 3D objects with surfaces. Techniques like B-spline surfaces and NURBS, evolving from Bézier's work, are widely used in engineering and computer graphics [1-3]. These methods enable the creation and analysis of complex structures in fields like CAD, computer graphics, and data visualization [4,5].

Uncertainty in data collection arises from measurement errors, sampling variability, and contextual influences, leading to a lack of confidence in the collected data [6]. It can be attributed to various factors, for example measurement errors, which are due to inaccuracies in instruments or human errors during data collection. Fuzzy set theory, proposed by Lotfi A. Zadeh in 1965, addressed the challenges of uncertainty in decision-making processes [7]. Fuzzy sets introduced membership degrees, enabling flexible reasoning and inference in domains such as control systems and pattern recognition. Jerry Mendel extended this concept with type-2 fuzzy sets in 1975, accommodating uncertainty about the membership function itself [8]. Intuitionistic fuzzy sets, proposed by Krassimir Atanassov in 1986, introduced non-membership degrees alongside membership, offering a more comprehensive representation of uncertainty [9]. Extensions like hesitant fuzzy sets by Torra and Narukawa in the early 2000s allow for multiple membership degrees, reflecting varying confidence levels in assigning membership [10]. Integrating fuzzy set theory with geometric modeling has garnered significant research interest. Anile *et al.* implemented fuzzy arithmetic for handling imprecise data, applied in environmental impact analysis [11]. Subsequently, Wahab *et al.* explored fuzzy Bézier and B-spline curves [12], extending to interpolation and surface modeling [13-19]. However, a gap remains in integrating hesitant fuzzy sets with geometric modeling, necessitating a more nuanced approach to uncertainty management [10].

This article aims to contribute to the development of a novel geometric model, namely hesitant fuzzy Bézier surface (HFBS) approximation model. It will begin by exploring the background and evolution of geometric modeling and hesitant fuzzy sets, followed by a review of pertinent literature. Section 2 will cover the fundamentals of hesitant fuzzy sets, providing detailed explanations of hesitant fuzzy set (HFS), hesitant fuzzy point (HFP), and hesitant fuzzy relation (HFR). Toward the end of this section, definitions of hesitant fuzzy point relation (HFPR) and hesitant fuzzy control net relation (HFCNR) will be introduced and utilized in the construction of the HFBS approximation model in Section 3. Section 4 will present visualizations of the output of the HFBS approximation model using an example of a biquartic Bézier surface. Section 5 will delve into visualization techniques, while Section 6 will summarize the key insights presented in the paper.

2. Preliminaries

This section will cover fundamental concepts such as HFS, HFP, and HFR. Following that, definitions for HFPR and HFCPR will also be defined. Torra introduced HFS in 2010 with the aim of addressing scenarios where elements display uncertainty or ambiguity regarding their membership in fuzzy sets, which is a common occurrence in real-world contexts [10]. Rather than assigning a single membership degree to elements, HFS allows for the representation of multiple membership degrees by various parties for each specific element.

Definition 1. [10, 20, 21] Let X be a fixed set. A hesitant fuzzy set (HFS) on X is in terms of a function applying to X and returns a subset of $[0,1]$. Mathematically, a HFS can be described as follows:

$$A = \{\langle x, h_A(x)\rangle | x \in X\} \quad (1)$$

where $h_A(x)$ is a set of some values in $[0,1]$, denotes the possible membership degrees of the element $x \in X$ to the set A. In that the case, $h = h_A(x)$ is called as a hesitant fuzzy element (HFE), while Θ represents the set of all HFEs.

As the HFEs can consist of more than one membership degrees, some special HFEs for $x \in X$ are given as follows [10]:

i. Empty set: $h = \{0\}$, denoted as O^* as simplification.
ii. Full set: $h = \{1\}$, denoted as I^*.
iii. Complete ignorance (all are possible), $h = [0,1] \triangleq U^*$.
iv. Nonsense set: $h = \emptyset^*$.

Definition 2. [22] Let $Y \subseteq X$ and $A = \{\langle x, h_A(x)\rangle | x \in X\}$ such that $h_A(x) \subseteq [0,1]$.Then, $A_Y \in HF(X)$ is defined as follows:

$$A_Y(x) = \begin{cases} A, & for\ x \in Y \\ \{0\}, & for\ x \in X \backslash Y \end{cases} \tag{2}$$

If Y is a singleton, namely $\{y\}$, then $A_{\{y\}}$ is called as a hesitant fuzzy point (HFP) or hesitant fuzzy singleton, which denoted by y_A.

Definition 3. [23, 24] Suppose X and Y are universal sets. A hesitant fuzzy subset $\mathcal{R}$ of $X \times Y$ is defined as a hesitant fuzzy relation (HFR) from X to Y, that is,

$$\mathcal{R} = \{\langle (x,y), h_{\mathcal{R}}(x,y)\rangle | (x,y) \in X \times Y\} \tag{3}$$

For all $(x,y) \in X \times Y$, $h_{\mathcal{R}}(x,y)$ is a set of values in [0,1], which are the possible membership degrees or the relations for respective x and y.

2.1 Hesitant Fuzzy Point Relation

Eq. (3) defined the HFR from X to Y, where X and Y are universal sets, while Eq. (2) describes the HFP. With these inspirations, the hesitant fuzzy point relation (HFPR) is defined to describe the relation between two HFPs.

Definition 4. Suppose X and Y are universal sets. Let $P \subseteq X$, $Q \subseteq Y$, $A = \{\langle x, h_A(x)\rangle | x \in X\}$ and $B = \{\langle y, h_B(x)\rangle | y \in Y\}$. Then by *Definition 1*, $h_A(x)$ and $h_B(x)$ are two sets of some possible values in $[0,1]$. Let P and Q are two HFPs, then by *Definition 2*, $A_{P=\{p\}} = \{\langle x, h_A(x)\rangle | x \in P\}$ and $B_{Q=\{q\}} = \{\langle y, h_B(x)\rangle | y \in Q\}$, where $\{p\}$ and $\{q\}$ are two singletons for sets P and Q respectively. A hesitant fuzzy subset $\mathcal{R}^*$ of $P \times Q$ is defined as a hesitant fuzzy point relation (HFPR) from X to Y, as follows:

$$\mathcal{R}^* = \{\langle (x_i, y_j), h_{R^*}(x_i, y_j)\rangle | x_i \in P_i, y_j \in Q_j, (x_i, y_j) \in P_i \times Q_j\} \tag{4}$$

For all $(x_i, y_j) \in P \times Q$, $h_{R^*}(x_i, y_j) \in h_A(x_i) \times h_B(y_j)$ are the set of values in $[0,1]$, which denotes the possible membership degrees of the HFPR for x_i and y_j.

2.2 Hesitant Fuzzy Control Net Relation

As discussed in the first section, control net (CN) plays an important role in the surface approximation model. CN are a set of points, which can affect the behaviours of a surface. Any slight change in the location of the CN will lead to a different approximation of geometry. In order to apply the concept of HFS in the Bézier surface approximation model, it is necessary to define hesitant fuzzy control net (HFCN) and hesitant fuzzy control net relation (HFCNR) as well based on the fundamental concept by Wahab *et al.* [16] and Eq. (4).

Definition 5. Suppose P represents the control net vertices of a Bézier surface such that

$$P = \{P_{i,j}\}_{i\in I, j\in J}, \tag{5}$$

where $I = \{0,1,2,\dots,n\}$, $J = \{0,1,2,\dots,m\}$and P is a set of $(n+1)(m+1)$ points denoting the coordinates of each control net vertices. Let $\mathcal{R}^*$ be a HFPR, then the hesitant fuzzy control net relation (HFCNR) is defined as a relation of $(n+1)(m+1)$ points with different number of choices as follows:

$$\mathcal{R}^*_{cp} = \left\{\langle\left(\mathcal{P}^{\mathcal{H}}_{i,j,q_{ij}}\right), h_{R^*}\left(\mathcal{P}^{\mathcal{H}}_{i,j,q_{ij}}\right)\rangle \,|\mathcal{P}^{\mathcal{H}}_{i,j,q_{ij}} \in \mathcal{P}^{\mathcal{H}}_{i,j,S_{ij}}, \left(\mathcal{P}^{\mathcal{H}}_{0,0,q_{00}},\dots,\mathcal{P}^{\mathcal{H}}_{0,m,q_{0m}},\dots,\mathcal{P}^{\mathcal{H}}_{n,m,q_{nm}}\right) \in \mathcal{P}^{\mathcal{H}}_{0,0,S_{00}} \times \dots \times \mathcal{P}^{\mathcal{H}}_{0,m,S_{0m}} \times \dots \mathcal{P}^{\mathcal{H}}_{n,m,S_{nm}}\right\}, \tag{6}$$

where $q_{ij} \in S_{iJ}$, $S_{ij} = \{0,1,2,\dots,s_{ij}\}$ and $\mathcal{P}^{\mathcal{H}}_{i,j,M_i}$ are the hesitant fuzzy control net vertices (HFCNVs) at (i,j)-th term. Noted that $q_{ij} = q_{kl}$ for $(i,j) \neq (k,l)$ is possible but not compulsory. For each $\mathcal{P}^{\mathcal{H}}_{i,j,S_{ij}}$, there are q_i number of possible memberships, therefore, the relation returns in a set of possible HFCNVs. The HFCNV at (i,j)-th term are defined as follows:

$$\begin{aligned}
&\mathcal{P}^{\mathcal{H}}_{0,0,q_{0,o}\in S_{0,0}} = \{\mathcal{P}^{\mathcal{H}}_{0,0,0}, \mathcal{P}^{\mathcal{H}}_{0,0,1}, \dots, \mathcal{P}^{\mathcal{H}}_{0,0,s_{0,0}}\}\\
&\vdots\\
&\mathcal{P}^{\mathcal{H}}_{0,m,q_{0,m}\in S_{0,m}} = \{\mathcal{P}^{\mathcal{H}}_{0,m,0}, \mathcal{P}^{\mathcal{H}}_{0,m,1}, \dots, \mathcal{P}^{\mathcal{H}}_{0,m,s_{0,m}}\}\\
&\vdots\\
&\mathcal{P}^{\mathcal{H}}_{n,m,q_{n,m}\in S_{n,m}} = \{\mathcal{P}^{\mathcal{H}}_{n,m,0}, \mathcal{P}^{\mathcal{H}}_{n,m,1}, \dots, \mathcal{P}^{\mathcal{H}}_{n,m,s_{n,m}}\}\quad .
\end{aligned} \tag{7}$$

In this case, HFCNVs is a set of $(n+1)(m+1)$ control net vertices, where each control points has s_i possibilities based on different hesitant fuzzy membership degrees assigned. Therefore, there are a variety number of possible HFCNRs, that is, a variety number of possible HFCNVs produced. The total number of HFCNRs, namely $N_{\mathcal{R}^*_{cnv}}$ is denoted by equation below:

$$N_{\mathcal{R}^*_{cnv}} = \prod_{i=0,j=0}^{n,m} s_{i,j}. \tag{8}$$

The HFCP can be redefined as

$$\mathcal{P}^{\mathcal{H}}_k = \left\{\{\mathcal{P}^{\mathcal{H}}_{0,0,q_{0,0}},\dots,\mathcal{P}^{\mathcal{H}}_{0,m,q_{0,m}},\dots,\mathcal{P}^{\mathcal{H}}_{n,m,q_{n,m}}\}|q_{i,j} \in S_{i,j}, S_{i,j} = \{0,1,2,\dots,s_{i,j}\}, \mathcal{P}^{\mathcal{H}}_{i,j,q_{i,j}} \in \mathcal{P}^{\mathcal{H}}_{i,j,S_{i,j}}\right\}. \tag{9}$$

for $N_{\mathcal{R}^*_{cnv}}$ possibilities, where $k = 1,2,,\dots N_{\mathcal{R}^*_{nv}}$. Therefore, the set of all HFCNs is described as follows:

$$\boldsymbol{\mathcal{P}^{\mathcal{H}}} = \left\{\mathcal{P}_k^{\mathcal{H}} | k = 1,2,\dots,N_{\mathcal{R}_{cnv}^*}\right\}. \tag{10}$$

3. Hesitant Fuzzy Bézier Surface Approximation Model

A Bézier surface approximation model constructs its surface by using Bernstein basis as its surface-blending functions [3]. As the HFCNR successfully defined, a hesitant fuzzy Bézier Surface (HFBS) approximation model is proposed and constructed, which applies the Bézier surface approximation model with the HFS from Eq. (1).

Definition 6. Let $\boldsymbol{\mathcal{P}^{\mathcal{H}}} = \left\{\mathcal{P}_k^{\mathcal{H}} | k = 1,2,\dots,N_{\mathcal{R}_{cnv}^*}\right\}$ be the set of all HFCNs such that $\mathcal{P}_k^{\mathcal{H}} = \left\{\left\{\mathcal{P}_{0,0,q_{0,0}}^{\mathcal{H}},\dots,\mathcal{P}_{0,m,q_{0,m}}^{\mathcal{H}},\dots,\mathcal{P}_{n,m,q_{n,m}}^{\mathcal{H}}\right\} | q_{i,j} \in S_{i,j}, S_{i,j} = \{0,1,\dots,s_{i,j}\}, \mathcal{P}_{i,j,q_{i,j}}^{\mathcal{H}} \in \mathcal{P}_{i,j,S_{i,j}}^{\mathcal{H}}\right\}$ is the possible HFCNs. Let $BS(u,w)$ as the surface position vector of a Bézier surface. A HFBS approximation model, namely $\boldsymbol{BS^{\mathcal{H}}}(u,w)$ is defined as follows:

$$\boldsymbol{BS^{\mathcal{H}}}(u,w) = \sum_{i=0}^{n}\sum_{j=0}^{m} N_{n,i}(u) \cdot \boldsymbol{\mathcal{P}^{\mathcal{H}}} \cdot M_{m,j}(w)\,, 0 \le u \le 1, o \le w \le 1 \tag{11}$$

where u and v are the parameters and the Bernstein basis, $N_{n,i}(u)$ and $M_{m,j}(w)$ are

$$N_{n,i}(u) = \binom{n}{i} u^i (1-u)^{n-i} \; with \; \binom{n}{i} = \frac{n!}{i!(n-i)!} \tag{12}$$

$$M_{m,j}(w) = \binom{m}{j} w^j (1-w)^{m-j} \; with \; \binom{m}{j} = \frac{m!}{j!(m-j)!} \tag{13}$$

Restrictions such as $(0)^0 \equiv 1$ and $0! \equiv 1$ are considered in this model. Since there are $N_{\mathcal{R}_{cn}^*}$ number of possible HFCNs, therefore, there are the same number of possible HFBSs equation as well. In other words, a $\boldsymbol{BS^{\mathcal{H}}}(u,w)$ is the set of all possible HFBSs, $BC_k^{\mathcal{H}}(t)$ such that

$$\boldsymbol{BS^{\mathcal{H}}}(u,w) = \left\{BS_k^{\mathcal{H}}(u,w) = \sum_{i=0}^{n}\sum_{j=0}^{m} N_{n,i}(u) \cdot \mathcal{P}_k^{\mathcal{H}} \cdot M_{m,j}(w) \,|\, k = 1,2,3,\dots,N_{\mathcal{R}_{cn}^*}\right\} \tag{14}$$

where $N_{\mathcal{R}_{cn}^*}$ number of HFBSs will be generated through the model based on *Definition 5*.

3.1 Properties of Hesitant Fuzzy Bézier Surface

A Bézier surface is generated based on its control net, which is formed by control net vertices. As the Bézier basis follows the Bernstein basis, Bézier surfaces will have the same properties [3]. That is, HFBS has the properties as follows:

i. The basis functions of HFBS are real.
ii. The degree of the HFBS in both parametric directions is one less than the number of HFCNVs in that particular direction.
iii. The continuity of the HFBS in both parametric directions is two less than he number of HFCNVs in that particular direction.
iv. The shape of HFBS will always follow the shape of HFCN.
v. The corner points of HFCNs are the only points which coincide with the resulting HFBS.
vi. HFBS generated will always be bound by its respective HFCN.

vii. HFBS does not exhibit variation-diminishing property, which is both undefined and unknown for bivariant surfaces.
viii. Under an affine transformation, HFBS is invariant.

4. Biquartic Hesitant Fuzzy Bézier Surface Visualization

In this section, a biquartic HFBS will be constructed and visualized. Suppose there is a square range with both length and width 4 meters, but the height varies. Two experts provide the height of the control net vertices to generate the surface in (x, y, z) coordinates, but there exists 2 points, which occurs conflict. The second expert, E_2 does not agree with the coordinates of those 2 points given by the first expert, E_1, and he provides a new coordinate for the points respectively. As they cannot persuade each other, hesitancy occurs. In this case, HFBS approximation model is applied to model the surface with uncertainty.

Let say the (x, y)-coordinates of the control net vertices remained crisp, while the z-coordinates is represented by the membership degrees given by different experts. Table 1 below shows the possible (x, y, z) coordinates for the HFCN.

Table 1
Possible (x, y, z) coordinates for HFCNs

(x, y)	z-coordinate		(x, y)	z-coordinate	
	E_1	E_2		E_1	E_2
(0,0)	0.8147	0.1379	(2,3)	0.9706	-
(0,1)	0.9134	-	(2,4)	0.8003	-
(0,2)	0.2785	-	(3,0)	0.9134	-
(0,3)	0.9649	-	(3,1)	0.2785	-
(0,4)	0.9572	0.2194	(3,2)	0.9649	-
(1,0)	0.9058	-	(3,3)	0.9572	-
(1,1)	0.6324	-	(3,4)	0.1419	-
(1,2)	0.5469	-	(4,0)	0.6324	-
(1,3)	0.1576	-	(4,1)	0.5469	-
(1,4)	0.4854	-	(4,2)	0.1576	-
(2,0)	0.1270	-	(4,3)	0.4854	-
(2,1)	0.0975	-	(4,4)	0.4218	-
(2,2)	0.9575	-			

By Eq. (8), the $N_{\mathcal{R}^*_{cn}}$ can be determined.

$$N_{\mathcal{R}^*_{cn}} = \prod_{i=0,j=0}^{4,4} s_{ij} = 2 \cdot 2 = 4,$$

$$\boldsymbol{\mathcal{P}^{\mathcal{H}}} = \{\mathcal{P}_1^{\mathcal{H}}, \mathcal{P}_2^{\mathcal{H}}, \mathcal{P}_3^{\mathcal{H}} \mathcal{P}_4^{\mathcal{H}}\},$$

$$\boldsymbol{BS^{\mathcal{H}}}(u, w) = \{BS_k^{\mathcal{H}}(u, w) = \sum_{i=0}^{n} \sum_{j=0}^{m} N_{4,i}(u) \cdot \mathcal{P}_k^{\mathcal{H}} \cdot M_{4,j}(w)\, | k = 1,2,3,4\}.$$

Therefore, there are 4 sets of $\boldsymbol{\mathcal{P}^{\mathcal{H}}}$ and its respective $\boldsymbol{BS^{\mathcal{H}}}(u, w)$ will be visualized by equations as above. The visualization of the possible HFBSs, $BS_k^{\mathcal{H}}(u, w)$ are shown. Figure 1 below shows all of the possible HFBSs in one space, which includes $BS_1^{\mathcal{H}}(u, w)$, $BS_2^{\mathcal{H}}(u, w)$, $BS_3^{\mathcal{H}}(u, w)$ and $BS_4^{\mathcal{H}}(u, w)$. Obviously, any slight change is HFCNVs in HFCN will affect the visualization of surface. The hesitancy can be observed from the surface which does not overlap each other. To check the properties of those HFBSs, it is necessary to separate them into particular figures.

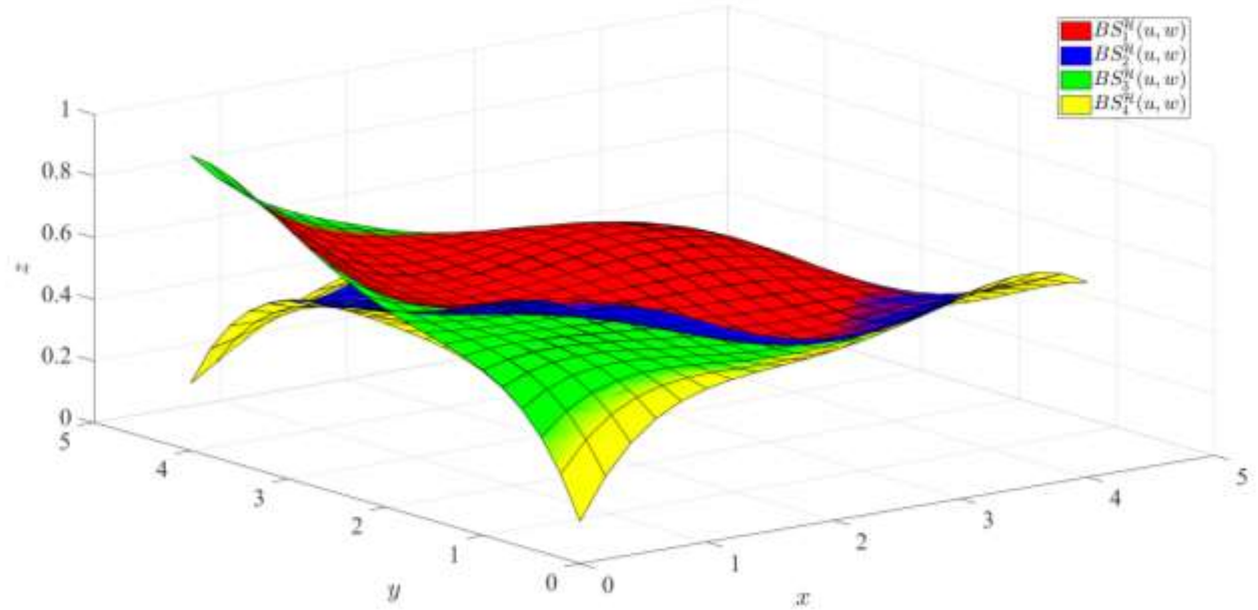

Fig. 1. Biquartic HFBSs in a same space, $BS_k^{\mathcal{H}}(u,w)$**,** $k = 1{,}2{,}3{,}4$

Figure 2, Figure 3, Figure 4 and Figure 5 below shows the visualization of different HFBSs based on their vertices, HFCNVs, which along with their largest convex hull of control net, HFCNs. Based on each figure, all of the bicubic HFBSs visualized achieve the properties as a HFBS. Obviously, every HFBS here are bounded by the largest convex hull of their HFCNs respectively. Apart from that, the corner points of the HFCNs in every HFBSs constructed coincide with the surface, while the shape of HFBSs follow the shape of respective HFCNs.

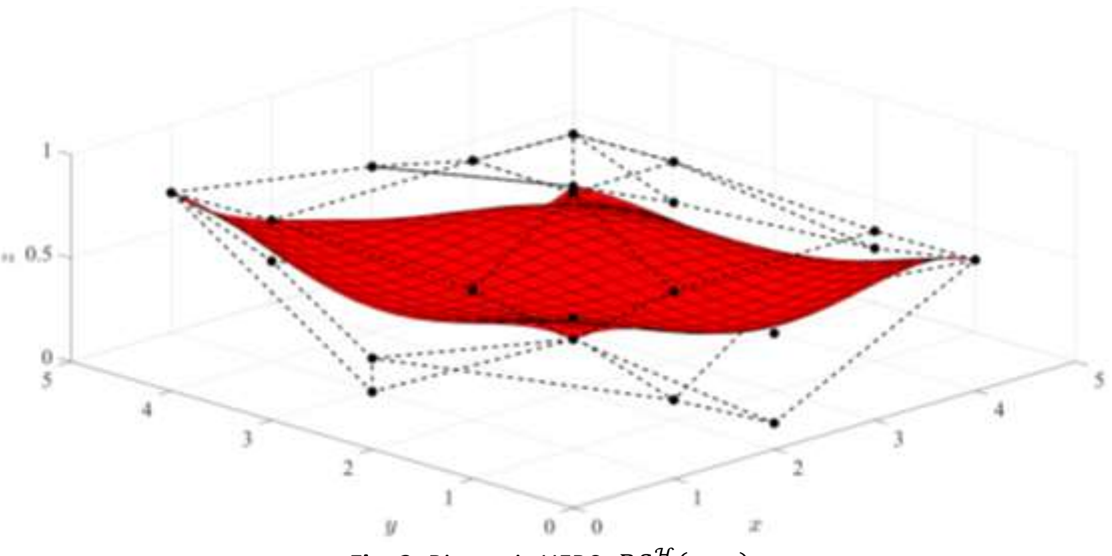

Fig. 2. Biquartic HFBS, $BS_1^{\mathcal{H}}(u,w)$

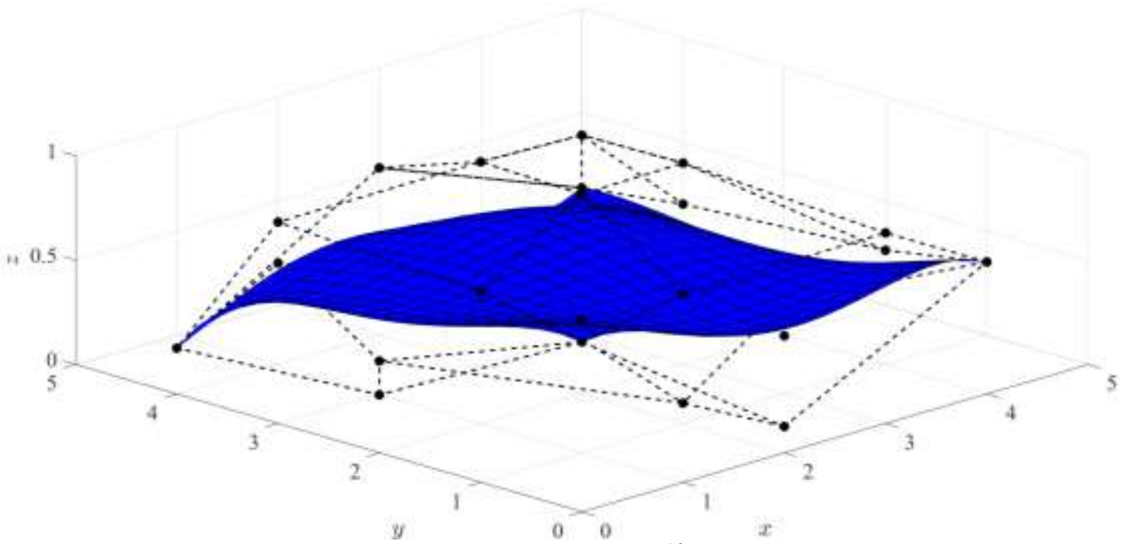

Fig. 3. Biquartic HFBS, $BS_2^{\mathcal{H}}(u,w)$

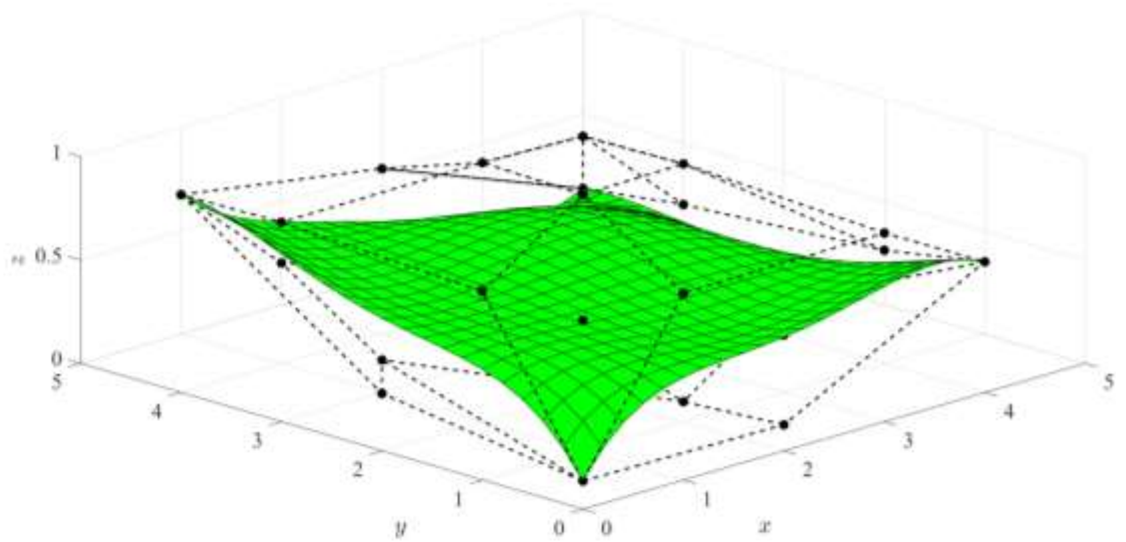

Fig. 4. Biquartic HFBS, $BS_3^{\mathcal{H}}(u,w)$

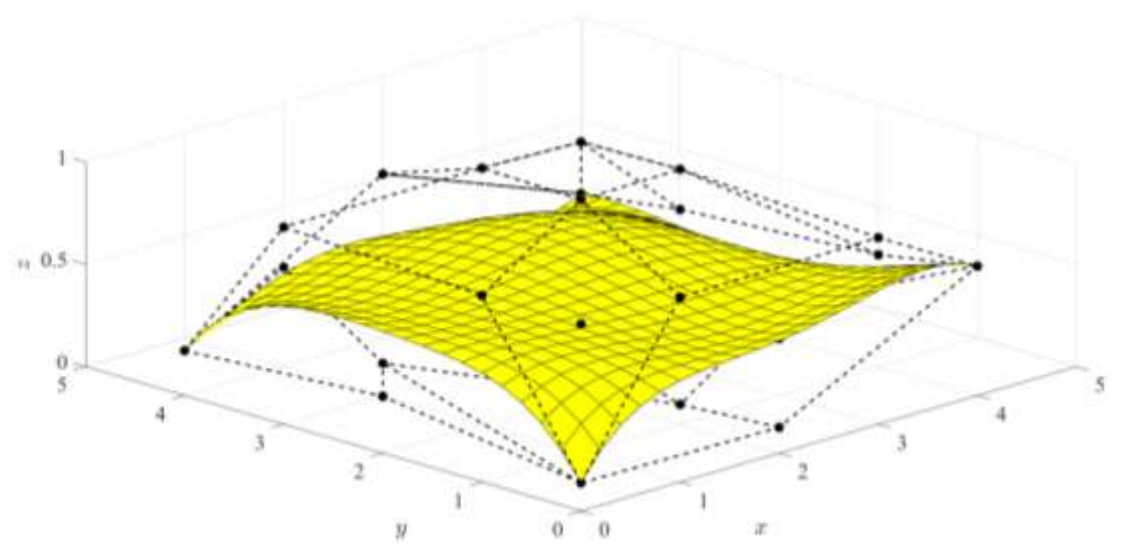

Fig. 5. Biquartic HFBS, $BS_4^{\mathcal{H}}(u,w)$

5. Discussion

In Section 4, an example of biquartic HFBS approximation model was successfully constructed. As result, the model produces 4 different HFBSs based on their HFCNs. In short, every biquartic HFBSs visualized are satisfying the properties of being a HFBS. Every HFBC visualized are different, which is caused by the hesitancy. A HFBS approximation model can visualize all surfaces based on the hesitancy value (membership degree). It provides a clear method to analyze the uncertainty and make conclusions.

The study introduces a novel approach to geometric modeling, focusing on hesitant fuzzy techniques, particularly in the context of Bézier surface approximation. It introduces models like HFCNR and HFBS for visualizing Bézier surfaces using hesitant fuzzy sets, offering a method to address uncertainty inherent in real-world scenarios. The mathematical framework of HFBS is defined and analyzed, along with the introduction and examination of its properties. This application of hesitant fuzzy sets provides a new avenue for handling uncertainty that is caused by hesitancy in geometric modeling, which is prevalent in practical applications. The study also suggests the potential extension of this approach to interpolation models, making it more applicable in real-world scenarios.

6. Conclusion

In short, the hesitant fuzzy Bézier surface (HFBS) model is successfully defined and constructed in this study. Besides, the properties of HFBS is introduced. HFBS approximation model has created a new way in surface modelling, which allows the model to visualize different surfaces based on hesitancy caused by different opinions (membership degrees). Every opinion involves in the construction of the model to visualize the surface. Nevertheless, the application of hesitancy in the model leads to a result of providing multiple output surface, which can only show the comparison but cannot contribute to making conclusions. Therefore, an appropriate justification method needs to be introduced as further study, which will justify the final result of the output. Apart from that, the study can be extended to an interpolation model, as an approximation model is the foundation of it. The interpolation model allows the application of data points rather than the control net vertices, which is more applicable in real-life problems.

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