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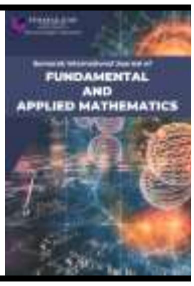

On A Simple Unifying Method for Higher Order Polynomial Decomposition

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ABSTRACT

The simple and unified method of higher order polynomial decomposition is proposed in this paper. The method increases the n-order polynomial into n+1-order and then divides the equation by second order polynomial with arbitrary coefficients. Some examples are given in order to demonstrate its simplicity and applicability for generating solution of nonlinear ODE. The main result is then concluded as a theorem.

1. Introduction

The methods for tackling nonlinear differential equations are rapidly developed analytically and numerically in recent years [1,2]. It is mainly motivated by the stricter requirements to implement environmentally friendly technologies which are less energy consumption, less or zero emission, zero waste, etc [3,4]. These goals seem can only be achieved faster by drastically changing our current technologies. As for example, implementing new advanced materials, harnessing a new source of energy that is like plasma whose nonlinearities cannot be neglected at high energy.

As one of the promising methods is the development of simplest equations to tackle such hard problems analytically. The method utilizes some simple nonlinear differential equations such as Bernoulli and Riccati into the original problems [5,6]. The algebraic polynomials with constant coefficients are then produced for constant coefficients of single nonlinear differential equations. The variable coefficients polynomial emerges for a system of equations, which both cases require root solution and technical difficulties rises for variable coefficients in the polynomials of higher order [7].

The methods of finding polynomial roots have long been developed with one important note which full radical solution does not exist for order five and higher, at least until now. Thus, the

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method for finding higher order roots is developed with polynomial decomposition into which the known radical solution existed. In short, the algorithm now exists for polynomial with arbitrary order [8]. Despite the various existing computer programs, we are still exposed to great difficulties applying the resulted complicated expressions from existing method to many nonlinear differential equations which, simpler procedure is demanded.

The main contribution of this work is to propose the simpler unifying method for decomposing polynomial of higher order. Here the n-order polynomial is firstly raised up into n+1-order and then decomposed by dividing by quadratic equation with arbitrary coefficients. The resulted decomposed coefficients are provided for exact and approximate cases which depend on the chosen quadratic coefficients. The procedure of providing arbitrary coefficients may be promising for nonlinear differential equations even recreational for learning junior students with sample of computer programs. In this work, some examples with matlab online programming are provided for decomposing 4th, 5th and 6th order polynomials as well as its applicability for solving nonlinear ODE.

2. The Main Results

Lemma 1. The 4th order polynomial equation can always be represented by the product of quadratic equation as in the following,

$$\left(c_1\lambda^3+c_2\lambda^2+c_3\lambda+c_4-1\right)\left[\left(b_1-a_4\right)\lambda+\left(b_2-a_5\right)\right]=0 \text{ or } \left(c_1\lambda^3+c_2\lambda^2+c_3\lambda+c_4-1\right)\left[\left(b_1-a_4\right)\lambda+\left(b_2-a_5\right)\right]\approx 0$$

Proof. Consider the 4th order polynomial equation,

$$a_1\lambda^4+a_2\lambda^3+a_3\lambda^2+a_4\lambda+a_5=0$$

Multiply the equation by, λ and rearranged as,

$$a_1\lambda^5+a_2\lambda^4+a_3\lambda^3+a_4\lambda^2+a_5\lambda=0 \text{ or } a_1\lambda^5+a_2\lambda^4+a_3\lambda^3=-a_4\lambda^2-a_5\lambda \text{ or}$$
$$a_1\lambda^5+a_2\lambda^4+a_3\lambda^3+b_1\lambda^2+b_2\lambda+b_3-a_6=\left(b_1-a_4\right)\lambda^2+\left(b_2-a_5\right)\lambda+\left(b_3-a_6\right) \quad (1)$$

The polynomial equation is rewritten as,

$$\left(c_1\lambda^3+c_2\lambda^2+c_3\lambda+c_4\right)\left[\left(b_1-a_4\right)\lambda^2+\left(b_2-a_5\right)\lambda+\left(b_3-a_6\right)\right]=\left(b_1-a_4\right)\lambda^2+\left(b_2-a_5\right)\lambda+\left(b_3-a_6\right) \text{ or}$$
$$c_1\left(b_1-a_4\right)\lambda^5+\left[c_2\left(b_1-a_4\right)+c_1\left(b_2-a_5\right)\right]\lambda^4+\left[c_3\left(b_1-a_4\right)+c_2\left(b_2-a_5\right)+c_1\left(b_3-a_6\right)\right]\lambda^3$$
$$+\left[c_4\left(b_1-a_4\right)+c_3\left(b_2-a_5\right)+c_2\left(b_3-a_6\right)\right]\lambda^2+\left[c_4\left(b_2-a_5\right)+c_3\left(b_3-a_6\right)\right]\lambda+c_4\left(b_3-a_6\right)= \quad (2)$$
$$\left(b_1-a_4\right)\lambda^2+\left(b_2-a_5\right)\lambda+\left(b_3-a_6\right)$$

The coefficient relations are,

$$
\begin{aligned}
&c_1(b_1-a_4)=a_1\\
&c_2(b_1-a_4)+c_1(b_2-a_5)=a_2\\
&c_3(b_1-a_4)+c_2(b_2-a_5)+c_1(b_3-a_6)=a_3\\
&c_4(b_1-a_4)+c_3(b_2-a_5)+c_2(b_3-a_6)=b_1\\
&c_4(b_2-a_5)+c_3(b_3-a_6)=b_2\\
&c_4(b_3-a_6)=(b_3-a_6)
\end{aligned}
\quad \text{or} \quad
\begin{aligned}
&c_1(b_1-a_4)=a_1\\
&c_2a_1=-c_1^2(b_2-a_5)+c_1a_2\\
&c_3a_1=-c_1c_2(b_2-a_5)-c_1^2(b_3-a_6)+c_1a_3\\
&c_4a_1=-c_1c_3(b_2-a_5)-c_1c_2(b_3-a_6)+a_1+c_1a_4\\
&c_4(b_2-a_5)+c_3(b_3-a_6)=(b_2-a_5)+a_5\\
&c_4(b_3-a_6)=(b_3-a_6)
\end{aligned}
\quad \text{or}
$$

$$
\begin{aligned}
&c_1(b_1-a_4)=a_1\\
&c_2a_1=-c_1^2(b_2-a_5)+c_1a_2\\
&c_3a_1^2=c_1^3(b_2-a_5)^2-c_1^2a_2(b_2-a_5)-c_1^2a_1(b_3-a_6)+c_1a_1a_3\\
&c_4a_1^3=-c_1^4(b_2-a_5)^3+c_1^3a_2(b_2-a_5)^2+2c_1^3a_1(b_2-a_5)(b_3-a_6)-c_1^2a_1a_3(b_2-a_5)\\
&-c_1^2a_1a_2(b_3-a_6)+a_1^3+c_1a_1^2a_4\\
&c_4(b_2-a_5)+c_3(b_3-a_6)=(b_2-a_5)+a_5\\
&c_4(b_3-a_6)=(b_3-a_6)
\end{aligned}
\tag{3}
$$

From the above relations, we have,

$$
c_3(b_3-a_6)^2=a_5(b_3-a_6)-a_6(b_2-a_5) \tag{4}
$$

Substituting into the original equations, the identical equations are produced as,

$$
\begin{aligned}
&c_1^4(b_2-a_5)^3-c_1^3a_2(b_2-a_5)^2-2c_1^3a_1(b_2-a_5)(b_3-a_6)+c_1^2a_1a_3(b_2-a_5)+c_1^2a_1a_2(b_3-a_6)-c_1a_1^2a_4=0\\
&c_1^4(b_2-a_5)^3-c_1^3a_2(b_2-a_5)^2-2c_1^3a_1(b_2-a_5)(b_3-a_6)+c_1^2a_1a_3(b_2-a_5)+c_1^2a_1a_2(b_3-a_6)-c_1a_1^2a_4=0
\end{aligned}
\tag{5}
$$

Thus, we choose performing Eq. (3) as follows,

$$
\begin{aligned}
&-c_1^4(b_2-a_5)^4+c_1^3a_2(b_2-a_5)^3+2c_1^3a_1(b_2-a_5)^2(b_3-a_6)-c_1^2a_1a_3(b_2-a_5)^2\\
&-c_1^2a_1a_2(b_2-a_5)(b_3-a_6)+c_1a_1^2a_4(b_2-a_5)+c_1^3a_1(b_2-a_5)^2(b_3-a_6)-c_1^2a_1a_2(b_2-a_5)(b_3-a_6) \qquad \text{or}\\
&-c_1^2a_1^2(b_3-a_6)^2+c_1a_1^2a_3(b_3-a_6)=a_1^3a_5\\
&c_1^4(b_2-a_5)^4-c_1^3a_2(b_2-a_5)^3-3c_1^3a_1(b_2-a_5)^2(b_3-a_6)+c_1^2a_1^2(b_3-a_6)^2+2c_1^2a_1a_2(b_2-a_5)(b_3-a_6)\\
&+c_1^2a_1a_3(b_2-a_5)^2-c_1a_1^2a_3(b_3-a_6)-c_1a_1^2a_4(b_2-a_5)+a_1^3a_5=0
\end{aligned}
\tag{6}
$$

It is important to note that the purpose is to decompose quartic equation while from equation (6) and (2), it will be fulfilled if we set $(b_3-a_6)=0$. Thus, (6) becomes,

$$
c_1^4(b_2-a_5)^4-c_1^3a_2(b_2-a_5)^3+c_1^2a_1a_3(b_2-a_5)^2-c_1a_1^2a_4(b_2-a_5)+a_1^3a_5=0 \tag{7}
$$

with (b_2-a_5) is taken as arbitrary values. The quartic roots are written as follows,

$$
c_1=j\big((b_2-a_5)\big) \tag{8}
$$

which will be determined, c_1. This will give the exact decomposition coefficients but, essentially does not reduce the polynomial order since it is cycling back into the 4th order for determining decomposed coefficients.

Fortunately, we can approximate (7) as quadratic equation below,

$$c_1^2 a_1 a_3 \left(b_2 - a_5\right)^2 - c_1 a_1^2 a_4 \left(b_2 - a_5\right) + a_1^3 a_5 = 0 \tag{9}$$

in the sense that one root of (9) will be very close to (7) and even the error is undetected by the computer programs if a_5 is sufficiently small. This will definitely reduce the polynomial order and it can be proceeded into the higher order which there is no radical solution. Thus, the 4th order polynomial is rewritten as,

$$\left(c_1\lambda^3 + c_2\lambda^2 + c_3\lambda + c_4 - 1\right)\left[\left(b_1 - a_4\right)\lambda + \left(b_2 - a_5\right)\right] = 0 \tag{10}$$

by applying (7) or approximated as,

$$\left(c_1\lambda^3 + c_2\lambda^2 + c_3\lambda + c_4 - 1\right)\left[\left(b_1 - a_4\right)\lambda + \left(b_2 - a_5\right)\right] \approx 0 \tag{11}$$

by (9). This proves lemma 1.

Example 1: Consider the following quartic equation,

$$15\lambda^4 - 6\lambda^3 - 54\lambda^2 - 268\lambda + 54 = 0 \tag{12}$$

Running through equation (7) with $\left(b_2 - a_5\right) = -6$, the program produces,

$$c_1 = \begin{pmatrix} 7.9573 \\ \text{-3.7210 + 4.7563i} \\ \text{-3.7210 - 4.7563i} \\ 0.4846 \end{pmatrix}, \left(b_1 - a_4\right) = \begin{pmatrix} 1.8851 \\ \text{-1.5305 - 1.9564i} \\ \text{-1.5305 + 1.9564i} \\ 30.9537 \end{pmatrix}, c_2 = \begin{pmatrix} 22.1448 \\ \text{-2.0224 -16.0611i} \\ \text{-2.0224 +16.0611i} \\ -0.0999 \end{pmatrix}, c_3 = \begin{pmatrix} 41.8391 \\ \text{46.9624 + 2.9345i} \\ \text{46.9624 - 2.9345i} \\ -1.7639 \end{pmatrix}$$

$$\text{and } c_4 = \begin{pmatrix} -8 \\ -8 \\ -8 \\ -8 \end{pmatrix} \tag{13}$$

which the polynomial coefficients are exactly decomposed in all rows. The following relations show that the results are reproducing the original polynomial,

Third row: $\left[\left(\text{-3.7210 - 4.7563i}\right)\lambda^3 + \left(\text{-2.0224 +16.0611i}\right)\lambda^2 + \left(\text{46.9624 - 2.9345i}\right)\lambda - 9\right]*$
$\left[\left(\text{-1.5305 + 1.9564i}\right)\lambda - 6\right] = 15\lambda^4 - 6\lambda^3 - 54\lambda^2 - 268\lambda + 54 = 0$

Fourth row: $\left(0.4846\lambda^3 - 0.0999\lambda^2 - 1.7639\lambda - 9\right)\left(30.9537\lambda - 6\right) = 15\lambda^4 - 6\lambda^3 - 54\lambda^2 - 268\lambda + 54 = 0$

Note that the function $\left(b_2 - a_5\right)$ is arbitrary, testing with, $\left(b_2 - a_5\right) = 18.23$, we obtain,

$c_1 = \begin{pmatrix} \text{-2.619} \\ \text{1.2247 + 1.5654i} \\ \text{1.2247 - 1.5654i} \\ \text{-0.1595} \end{pmatrix}$, $(b_1 - a_4) = \begin{pmatrix} \text{-5.7274} \\ \text{4.6502 - 5.9441i} \\ \text{4.6502 + 5.9441i} \\ \text{-94.0477} \end{pmatrix}$, $c_2 = \begin{pmatrix} \text{-7.2885} \\ \text{0.6656 - 5.2861i} \\ \text{0.6656 + 5.2861i} \\ \text{0.0329} \end{pmatrix}$, $c_3 = \begin{pmatrix} \text{-13.7704} \\ \text{-15.4566 + 0.9658i} \\ \text{-15.4566 - 0.9658i} \\ \text{0.5806} \end{pmatrix}$

and $c_4 = \begin{pmatrix} 3.9622 \\ 3.9622 \\ 3.9622 \\ 3.9622 \end{pmatrix}$ (14)

Performing the first and second rows and they give,

First row: $\left(-2.619\lambda^3 - 7.2885\lambda^2 - 13.7704\lambda + 2.9622\right)\left(\text{-5.7274}\lambda + 18.23\right) = 15\lambda^4 - 6\lambda^3 - 54\lambda^2 - 268\lambda + 54 = 0$

Second row: $\left[\left(\text{1.2247 + 1.5654i}\right)\lambda^3 + \left(\text{0.6656 - 5.2861i}\right)\lambda^2 + \left(\text{-15.4566 + 0.9658i}\right)\lambda - 2.9622\right]*$
$\left[\left(\text{4.6502 - 5.9441i}\right)\lambda + 18.23\right] = 15\lambda^4 - 6\lambda^3 - 54\lambda^2 - 268\lambda + 54 = 0$

and are also the exact results. The reader may test various numbers for $(b_2 - a_5)$ and will get the exact results by applying (7).

The step now is fully utilizing lemma 1 which will implement reduction of order by relation (9) and $(b_2 - a_5) = 5.67$, as in the following,

$c_1 = \begin{pmatrix} 13.6425 \\ \text{-0.5130} \end{pmatrix}$, $(b_1 - a_4) = \begin{pmatrix} 1.0995 \\ \text{-29.2395} \end{pmatrix}$, $c_2 = \begin{pmatrix} \text{-75.8099} \\ 0.1057 \end{pmatrix}$, $c_3 = \begin{pmatrix} 341.8294 \\ 1.8673 \end{pmatrix}$ and $c_4 = \begin{pmatrix} \text{-2005.5} \\ 10.5 \end{pmatrix}$ (15)

which will produce the quartic as follows,

First row: $\left(13.6425\lambda^3 - 75.8099\lambda^2 + 341.8294\lambda - 2006.5\right)\left(1.0995\lambda + 5.67\right) = 15\lambda^4 - 6\lambda^3 - 54\lambda^2 - 268\lambda - 11377$
$\neq 15\lambda^4 - 6\lambda^3 - 54\lambda^2 - 268\lambda + 54 = 0$

Second row: $\left(-0.513\lambda^3 + 0.1057\lambda^2 + 1.8673\lambda + 9.5\right)\left(\text{-29.2395}\lambda + 5.67\right) = 15\lambda^4 - 6\lambda^3 - 54\lambda^2 - 268\lambda + 54 = 0$

The second row almost shows an exact result for the reduction of (7) into the quadratic equation (9) which its numerical error is undetected. Trying other values for $(b_2 - a_5)$ and different decomposed coefficients but the same polynomial coefficients are obtained.

Lemma 2. The 5th order polynomial,

$$a_1\lambda^5 + a_2\lambda^4 + a_3\lambda^3 + a_4\lambda^2 + a_5\lambda + a_6 = 0$$

where, a_6 is sufficiently small, can be approximately factorized as,

$\left(c_1\lambda^4 + c_2\lambda^3 + c_3\lambda^2 + c_4\lambda + c_4 - 1\right)\left[\left(b_1 - a_5\right)\lambda + \left(b_2 - a_6\right)\right] = 0$ or

$$\left(c_1\lambda^4+c_2\lambda^3+c_3\lambda^2+c_4\lambda+c_4-1\right)\left[\left(b_1-a_5\right)\lambda+\left(b_2-a_6\right)\right]\approx 0$$

Proof. Consider the quintic polynomial as follows,

$$a_1\lambda^5+a_2\lambda^4+a_3\lambda^3+a_4\lambda^2+a_5\lambda+a_6=0$$

Multiply the above equation by λ,

$$a_1\lambda^6+a_2\lambda^5+a_3\lambda^4+a_4\lambda^3+a_5\lambda^2+a_6\lambda=0 \qquad \text{or}$$
$$a_1\lambda^6+a_2\lambda^5+a_3\lambda^4+a_4\lambda^3+b_1\lambda^2+b_2\lambda+b_3=\left(b_1-a_5\right)\lambda^2+\left(b_2-a_6\right)\lambda+\left(b_3-a_7\right)=0 \tag{16}$$

with, a_6 is small. The polynomial equation is rewritten as,

$$\left(c_1\lambda^4+c_2\lambda^3+c_3\lambda^2+c_4\lambda+c_5\right)\left[\left(b_1-a_5\right)\lambda^2+\left(b_2-a_6\right)\lambda+\left(b_3-a_7\right)\right]=\left(b_1-a_5\right)\lambda^2+\left(b_2-a_6\right)\lambda+\left(b_3-a_7\right) \qquad \text{or}$$
$$\begin{aligned}&c_1\left(b_1-a_5\right)\lambda^6+\left[c_2\left(b_1-a_5\right)+c_1\left(b_2-a_6\right)\right]\lambda^5+\left[c_3\left(b_1-a_5\right)+c_2\left(b_2-a_6\right)+c_1\left(b_3-a_7\right)\right]\lambda^4\\&+\left[c_4\left(b_1-a_5\right)+c_3\left(b_2-a_6\right)+c_2\left(b_3-a_7\right)\right]\lambda^3+\left[c_5\left(b_1-a_5\right)+c_4\left(b_2-a_6\right)+c_3\left(b_3-a_7\right)\right]\lambda^2\\&+\left[c_5\left(b_2-a_6\right)+c_4\left(b_3-a_7\right)\right]\lambda+c_5\left(b_3-a_7\right)=\left(b_1-a_5\right)\lambda^2+\left(b_2-a_6\right)\lambda+\left(b_3-a_7\right)\end{aligned} \tag{17}$$

The coefficient relations are

$$\begin{aligned}&c_1\left(b_1-a_5\right)=a_1\\&c_2\left(b_1-a_5\right)+c_1\left(b_2-a_6\right)=a_2\\&c_3\left(b_1-a_5\right)+c_2\left(b_2-a_6\right)+c_1\left(b_3-a_7\right)=a_3\\&c_4\left(b_1-a_5\right)+c_3\left(b_2-a_6\right)+c_2\left(b_3-a_7\right)=a_4\\&c_5\left(b_1-a_5\right)+c_4\left(b_2-a_6\right)+c_3\left(b_3-a_7\right)=b_1\\&c_5\left(b_2-a_6\right)+c_4\left(b_3-a_7\right)=b_2\\&c_5\left(b_3-a_7\right)=\left(b_3-a_7\right)\end{aligned} \qquad \text{or} \qquad \begin{aligned}&c_1\left(b_1-a_5\right)=a_1\\&c_2a_1=-c_1^2\left(b_2-a_6\right)+c_1a_2\\&c_3a_1=-c_1c_2\left(b_2-a_6\right)-c_1^2\left(b_3-a_7\right)+c_1a_3\\&c_4a_1=-c_1c_3\left(b_2-a_6\right)-c_1c_2\left(b_3-a_7\right)+c_1a_4\\&c_5a_1=-c_1c_4\left(b_2-a_6\right)-c_1c_3\left(b_3-a_7\right)+a_1+c_1a_5\\&c_5\left(b_2-a_6\right)+c_4\left(b_3-a_7\right)=\left(b_2-a_6\right)+a_6\\&c_5\left(b_3-a_7\right)=\left(b_3-a_7\right)\end{aligned} \qquad \text{or}$$

$$\begin{aligned}&c_1\left(b_1-a_5\right)=a_1\\&c_2a_1=-c_1^2\left(b_2-a_6\right)+c_1a_2\\&c_3a_1^2=c_1^3\left(b_2-a_6\right)^2-c_1^2a_2\left(b_2-a_6\right)-c_1^2a_1\left(b_3-a_7\right)+c_1a_1a_3\\&c_4a_1^3=-c_1^4\left(b_2-a_6\right)^3+c_1^3a_2\left(b_2-a_6\right)^2+c_1^3a_1\left(b_2-a_6\right)\left(b_3-a_7\right)-c_1^2a_1a_3\left(b_2-a_6\right)\\&+c_1^3a_1\left(b_2-a_6\right)\left(b_3-a_7\right)-c_1^2a_1a_2\left(b_3-a_7\right)+c_1a_1^2a_4\\&c_5a_1^4=c_1^5\left(b_2-a_6\right)^4-c_1^4a_2\left(b_2-a_6\right)^3-3c_1^4a_1\left(b_2-a_6\right)^2\left(b_3-a_7\right)+c_1^3a_1a_3\left(b_2-a_6\right)^2\\&+2c_1^3a_1a_2\left(b_2-a_6\right)\left(b_3-a_7\right)-c_1^2a_1^2a_4\left(b_2-a_6\right)+c_1^3a_1^2\left(b_3-a_7\right)^2-c_1^2a_1^2a_3\left(b_3-a_7\right)+a_1^4+c_1a_1^3a_5\\&c_5\left(b_2-a_6\right)+c_4\left(b_3-a_7\right)=\left(b_2-a_6\right)+a_6\\&c_5\left(b_3-a_7\right)=\left(b_3-a_7\right)\end{aligned} \tag{18}$$

In this case, we leave to the readers to verify that there is identical equation in (18), the following expression is then producing,

$$c_1^5(b_2-a_6)^5 - c_1^4a_2(b_2-a_6)^4 - 4c_1^4a_1(b_2-a_6)^3(b_3-a_7) + c_1^3a_1a_3(b_2-a_6)^3 + 2c_1^3a_1a_2(b_2-a_6)^2(b_3-a_7)$$
$$+3c_1^3a_1^2(b_2-a_6)(b_3-a_7)^2 - c_1^2a_1^2a_2(b_3-a_7)^2 - c_1^2a_1^2a_4(b_2-a_6)^2 - c_1^2a_1^2a_3(b_2-a_6)(b_3-a_7) + c_1a_1^3a_5(b_2-a_6) \text{ or}$$
$$+c_1^3a_1a_2(b_2-a_6)^2(b_3-a_7) - c_1^2a_1^2a_3(b_2-a_6)(b_3-a_7) + c_1a_1^3a_4(b_3-a_7) = a_1^4a_6$$
$$c_1^5(b_2-a_6)^5 - c_1^4a_2(b_2-a_6)^4 - 4c_1^4a_1(b_2-a_6)^3(b_3-a_7) + c_1^3a_1a_3(b_2-a_6)^3 + 3c_1^3a_1a_2(b_2-a_6)^2(b_3-a_7)$$
$$+3c_1^3a_1^2(b_2-a_6)(b_3-a_7)^2 - c_1^2a_1^2a_2(b_3-a_7)^2 - c_1^2a_1^2a_4(b_2-a_6)^2 - 2c_1^2a_1^2a_3(b_2-a_6)(b_3-a_7) \quad (19)$$
$$+c_1a_1^3a_4(b_3-a_7) + c_1a_1^3a_5(b_2-a_6) - a_1^4a_6 = 0$$

Take one quintic root with $(b_3-a_7)=0$ as follows,

$$c_1 = j\left((b_2-a_6)\right) \quad (20)$$

which then produces the exact decomposition coefficients with arbitrary (b_2-a_6).

The order reduction is in the following relation,

$$c_1^3a_1a_3(b_2-a_6)^3 - c_1^2a_1^2a_4(b_2-a_6)^2 + c_1a_1^3a_5(b_2-a_6) - a_1^4a_6 = 0 \quad (21)$$

and c_1 will be well-approximated in the sense that the decomposed coefficients are very closely reproduce the original polynomial. Thus, the 5th order polynomial is decomposed as,

$$\left(c_1\lambda^4 + c_2\lambda^3 + c_3\lambda^2 + c_4\lambda + c_5 - 1\right)\left[(b_1-a_5)\lambda + (b_2-a_6)\right] = 0 \text{ or}$$
$$\left(c_1\lambda^4 + c_2\lambda^3 + c_3\lambda^2 + c_4\lambda + c_5 - 1\right)\left[(b_1-a_5)\lambda + (b_2-a_6)\right] \approx 0 \quad (22)$$

This proves lemma 2.

Example 2: Consider the quintic equation as follows,

$$2\lambda^5 - 3\lambda^4 - 23\lambda^3 + 45\lambda^2 - 7\lambda - 6 = 0 \quad (23)$$

Running through with $(b_2-a_6)=-2$, the program produces,

$$c_1 = \begin{pmatrix} -3.5616 \\ 3 \\ 1.7808 \\ 0.5616 \\ -0.2808 \end{pmatrix},\ (b_1-a_5) = \begin{pmatrix} -0.5616 \\ 0.6667 \\ 1.1231 \\ 3.5616 \\ -7.1231 \end{pmatrix},\ c_2 = \begin{pmatrix} 18.0270 \\ 4.5 \\ 0.5 \\ -0.5270 \\ 0.5 \end{pmatrix},\ c_3 = \begin{pmatrix} -23.2462 \\ -21 \\ -19.5885 \\ -6.7538 \\ 3.0885 \end{pmatrix},\ c_4 = \begin{pmatrix} 2.6577 \\ 4.5 \\ 5.1847 \\ 8.8423 \\ -7.1847 \end{pmatrix} \text{ and } c_5 = \begin{pmatrix} 4 \\ 4 \\ 4 \\ 4 \\ 4 \end{pmatrix} \quad (24)$$

which also exactly matches (23) in all rows and their roots can be performed by known radical solution. The sample of calculations are,

Second row: $\left(3\lambda^4 + 4.5\lambda^3 - 21\lambda^2 + 4.5\lambda + 3\right)(0.6667\lambda - 2) = 2\lambda^5 - 3\lambda^4 - 23\lambda^3 + 45\lambda^2 - 7\lambda - 6 = 0$

Third row: $\left(1.7808\lambda^4 + 0.5\lambda^3 - 19.5885\lambda^2 + 5.1847\lambda + 3\right)(1.1231\lambda - 2) = 2\lambda^5 - 3\lambda^4 - 23\lambda^3 + 45\lambda^2 - 7\lambda - 6 = 0$

Fourth row: $\left(0.5616\lambda^4 - 0.527\lambda^3 - 6.7538\lambda^2 + 8.8423\lambda + 3\right)\left(3.5616\lambda - 2\right) = 2\lambda^5 - 3\lambda^4 - 23\lambda^3 + 45\lambda^2 - 7\lambda - 6 = 0$

The reader can try any number for $(b_2 - a_6)$ and will exactly reproduce the original polynomial from the obtained decomposed coefficients.

Applying order reduction from relation (21) and with $(b_2 - a_6) = -42.93$ will generate the following decomposed coefficients,

$$c_1 = \begin{pmatrix} 0.0784 \\ 0.0258 \\ -0.0131 \end{pmatrix},\ (b_1 - a_5) = \begin{pmatrix} 25.4967 \\ 77.6364 \\ -153.2181 \end{pmatrix},\ c_2 = \begin{pmatrix} 0.0144 \\ -0.0244 \\ 0.0232 \end{pmatrix},\ c_3 = \begin{pmatrix} -0.8778 \\ -0.3097 \\ 0.1436 \end{pmatrix},\ c_4 = \begin{pmatrix} 0.2869 \\ 0.4083 \\ -0.3339 \end{pmatrix} \text{ and } c_5 = \begin{pmatrix} 1.2086 \\ 1.1356 \\ 1.1393 \end{pmatrix} \quad (25)$$

The resulted polynomials will be,

First row: $\left(0.0784\lambda^4 + 0.0144\lambda^3 - 0.8778\lambda^2 + 0.2869\lambda + 0.2086\right)\left(25.4967\lambda - 42.93\right)$
$= 2\lambda^5 - 3\lambda^4 - 23\lambda^3 + 45\lambda^2 - 7\lambda - 8.9537 \approx 2\lambda^5 - 3\lambda^4 - 23\lambda^3 + 45\lambda^2 - 7\lambda - 6 = 0$

Second row: $\left(0.0258\lambda^4 - 0.0244\lambda^3 - 0.3097\lambda^2 + 0.4083\lambda + 0.1356\right)\left(77.6364\lambda - 42.93\right)$
$= 2\lambda^5 - 3\lambda^4 - 23\lambda^3 + 45\lambda^2 - 7\lambda - 5.8229 \approx 2\lambda^5 - 3\lambda^4 - 23\lambda^3 + 45\lambda^2 - 7\lambda - 6 = 0$

Third row: $\left(-0.0131\lambda^4 + 0.0232\lambda^3 + 0.1436\lambda^2 - 0.3339\lambda + 0.1393\right)\left(-153.2181\lambda - 42.93\right)$
$= 2\lambda^5 - 3\lambda^4 - 23\lambda^3 + 45\lambda^2 - 7\lambda - 5.9781 \approx 2\lambda^5 - 3\lambda^4 - 23\lambda^3 + 45\lambda^2 - 7\lambda - 6 = 0$

Lemma 3. The 6th order polynomial,

$$a_1\lambda^6 + a_2\lambda^5 + a_3\lambda^4 + a_4\lambda^3 + a_5\lambda^2 + a_6\lambda + a_7 = 0$$

can be decomposed as follows,

$\left(c_1\lambda^5 + c_2\lambda^4 + c_3\lambda^3 + c_4\lambda^2 + c_5\lambda + c_6 - 1\right)\left[(b_1 - a_6)\lambda + (b_2 - a_7)\right] = 0$ or
$\left(c_1\lambda^5 + c_2\lambda^4 + c_3\lambda^3 + c_4\lambda^2 + c_5\lambda + c_6 - 1\right)\left[(b_1 - a_6)\lambda + (b_2 - a_7)\right] \approx 0$

Proof. Multiplying by λ, the polynomial is rewritten as,

$a_1\lambda^7 + a_2\lambda^6 + a_3\lambda^5 + a_4\lambda^4 + a_5\lambda^3 = -a_6\lambda^2 - a_7\lambda$ or
$$a_1\lambda^7 + a_2\lambda^6 + a_3\lambda^5 + a_4\lambda^4 + a_5\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 - a_8 = (b_1 - a_6)\lambda^2 + (b_2 - a_7)\lambda + (b_3 - a_8) \quad (26)$$

The equation also rewritten as,

$\left(c_1\lambda^5 + c_2\lambda^4 + c_3\lambda^3 + c_4\lambda^2 + c_5\lambda + c_6\right)\left[(b_1 - a_6)\lambda^2 + (b_2 - a_7)\lambda + (b_3 - a_8)\right] = (b_1 - a_6)\lambda^2 + (b_2 - a_7)\lambda + (b_3 - a_8)$ or

$$\begin{aligned}
&c_1(b_1-a_6)\lambda^7+c_2(b_1-a_6)\lambda^6+c_3(b_1-a_6)\lambda^5+c_4(b_1-a_6)\lambda^4+c_5(b_1-a_6)\lambda^3+c_6(b_1-a_6)\lambda^2\\
&+c_1(b_2-a_7)\lambda^6+c_2(b_2-a_7)\lambda^5+c_3(b_2-a_7)\lambda^4+c_4(b_2-a_7)\lambda^3+c_5(b_2-a_7)\lambda^2+c_6\lambda(b_2-a_7)\\
&+c_1(b_3-a_8)\lambda^5+c_2(b_3-a_8)\lambda^4+c_3(b_3-a_8)\lambda^3+c_4(b_3-a_8)\lambda^2+c_5(b_3-a_8)\lambda+c_6(b_3-a_8)\\
&=\left[(b_1-a_6)\lambda^2+(b_2-a_7)\lambda+(b_3-a_8)\right]
\end{aligned}$$

or

$$\begin{aligned}
&c_1(b_1-a_6)\lambda^7+\left[c_2(b_1-a_6)+c_1(b_2-a_7)\right]\lambda^6+\left[c_3(b_1-a_6)+c_2(b_2-a_7)+c_1(b_3-a_8)\right]\lambda^5\\
&+\left[c_4(b_1-a_6)+c_3(b_2-a_7)+c_2(b_3-a_8)\right]\lambda^4+\left[c_5(b_1-a_6)+c_4(b_2-a_7)+c_3(b_3-a_8)\right]\lambda^3\\
&+\left[c_6(b_1-a_6)+c_5(b_2-a_7)+c_4(b_3-a_8)\right]\lambda^2+\left[c_6(b_2-a_7)+c_5(b_3-a_8)\right]\lambda+c_6(b_3-a_8)\\
&=\left[(b_1-a_6)\lambda^2+(b_2-a_7)\lambda+(b_3-a_8)\right]
\end{aligned} \tag{27}$$

The coefficient relations are

$$\begin{aligned}
&c_1(b_1-a_6)=a_1\\
&c_2(b_1-a_6)+c_1(b_2-a_7)=a_2\\
&c_3(b_1-a_6)+c_2(b_2-a_7)+c_1(b_3-a_8)=a_3\\
&c_4(b_1-a_6)+c_3(b_2-a_7)+c_2(b_3-a_8)=a_4\\
&c_5(b_1-a_6)+c_4(b_2-a_7)+c_3(b_3-a_8)=a_5\\
&c_6(b_1-a_6)+c_5(b_2-a_7)+c_4(b_3-a_8)=b_1\\
&c_6(b_2-a_7)+c_5(b_3-a_8)=b_2\\
&c_6(b_3-a_8)=(b_3-a_8)
\end{aligned}$$

or

$$\begin{aligned}
&c_1(b_1-a_6)=a_1\\
&c_2a_1=-c_1^2(b_2-a_7)+c_1a_2\\
&c_3a_1=-c_1c_2(b_2-a_7)-c_1^2(b_3-a_8)+c_1a_3\\
&c_4a_1=-c_1c_3(b_2-a_7)-c_1c_2(b_3-a_8)+c_1a_4\\
&c_5a_1=-c_1c_4(b_2-a_7)-c_1c_3(b_3-a_8)+c_1a_5\\
&c_6a_1=-c_1c_5(b_2-a_7)-c_1c_4(b_3-a_8)+a_1+c_1a_6\\
&c_6(b_2-a_7)+c_5(b_3-a_8)=(b_2-a_7)+a_7\\
&c_6(b_3-a_8)=(b_3-a_8)
\end{aligned}$$

or

$$\begin{aligned}
&c_1(b_1-a_6)=a_1\\
&c_2a_1=-c_1^2(b_2-a_7)+c_1a_2\\
&c_3a_1^2=c_1^3(b_2-a_7)^2-c_1^2a_2(b_2-a_7)-c_1^2a_1(b_3-a_8)+c_1a_1a_3\\
&c_4a_1^3=-c_1^4(b_2-a_7)^3+c_1^3a_2(b_2-a_7)^2+2c_1^3a_1(b_2-a_7)(b_3-a_8)-c_1^2a_1a_3(b_2-a_7)-c_1^2a_1a_2(b_3-a_8)+c_1a_1^2a_4\\
&c_5a_1^4=c_1^5(b_2-a_7)^4-c_1^4a_2(b_2-a_7)^3-3c_1^4a_1(b_2-a_7)^2(b_3-a_8)+c_1^3a_1a_3(b_2-a_7)^2+2c_1^3a_1a_2(b_2-a_7)(b_3-a_8)\\
&+c_1^3a_1^2(b_3-a_8)^2-c_1^2a_1^2a_4(b_2-a_7)-c_1^2a_1^2a_3(b_3-a_8)+c_1a_1^3a_5\\
&c_6a_1^5=-c_1^6(b_2-a_7)^5+c_1^5a_2(b_2-a_7)^4+4c_1^5a_1(b_2-a_7)^3(b_3-a_8)-3c_1^4a_1a_2(b_2-a_7)^2(b_3-a_8)-c_1^4a_1a_3(b_2-a_7)^3\\
&-3c_1^4a_1^2(b_2-a_7)(b_3-a_8)^2+c_1^3a_1^2a_4(b_2-a_7)^2+c_1^3a_1^2a_2(b_3-a_8)^2+2c_1^3a_1^2a_3(b_2-a_7)(b_3-a_8)\\
&-c_1^2a_1^3a_5(b_2-a_7)-c_1^2a_1^3a_4(b_3-a_8)+a_1^5+c_1a_1^4a_6\\
&c_6(b_2-a_7)+c_5(b_3-a_8)=(b_2-a_7)+a_7\\
&c_6(b_3-a_8)=(b_3-a_8)
\end{aligned} \tag{28}$$

Next, Eq. (28) is rearranged as,

$$-c_1^6(b_2-a_7)^6+c_1^5a_2(b_2-a_7)^5+4c_1^5a_1(b_2-a_7)^4(b_3-a_8)-3c_1^4a_1a_2(b_2-a_7)^3(b_3-a_8)-c_1^4a_1a_3(b_2-a_7)^4$$
$$-3c_1^4a_1^2(b_2-a_7)^2(b_3-a_8)^2+c_1^3a_1^2a_4(b_2-a_7)^3+c_1^3a_1^2a_2(b_2-a_7)(b_3-a_8)^2+2c_1^3a_1^2a_3(b_2-a_7)^2(b_3-a_8)$$
$$-c_1^2a_1^3a_5(b_2-a_7)^2-c_1^2a_1^3a_4(b_2-a_7)(b_3-a_8)+c_1a_1^4a_6(b_2-a_7)$$
$$+c_1^5a_1(b_2-a_7)^4(b_3-a_8)-c_1^4a_1a_2(b_2-a_7)^3(b_3-a_8)-3c_1^4a_1^2(b_2-a_7)^2(b_3-a_8)^2+c_1^3a_1^2a_3(b_2-a_7)^2(b_3-a_8)$$
$$+2c_1^3a_1^2a_2(b_2-a_7)(b_3-a_8)^2+c_1^3a_1^3(b_3-a_8)^3-c_1^2a_1^3a_4(b_2-a_7)(b_3-a_8)-c_1^2a_1^3a_3(b_3-a_8)^2$$
$$+c_1a_1^4a_5(b_3-a_8)=a_1^5a_7$$

or

$$c_1^6(b_2-a_7)^6-c_1^5a_2(b_2-a_7)^5-5c_1^5a_1(b_2-a_7)^4(b_3-a_8)+4c_1^4a_1a_2(b_2-a_7)^3(b_3-a_8)+c_1^4a_1a_3(b_2-a_7)^4$$
$$+6c_1^4a_1^2(b_2-a_7)^2(b_3-a_8)^2-c_1^3a_1^2a_4(b_2-a_7)^3-3c_1^3a_1^2a_2(b_2-a_7)(b_3-a_8)^2-3c_1^3a_1^2a_3(b_2-a_7)^2(b_3-a_8)$$
$$-c_1^3a_1^3(b_3-a_8)^3+c_1^2a_1^3a_5(b_2-a_7)^2+2c_1^2a_1^3a_4(b_2-a_7)(b_3-a_8)+c_1^2a_1^3a_3(b_3-a_8)^2-c_1a_1^4a_5(b_3-a_8)$$
$$-c_1a_1^4a_6(b_2-a_7)+a_1^5a_7=0 \tag{29}$$

Take one root with $(b_3-a_8)=0$ as follows,

$$c_1=j\left((b_2-a_7)\right) \tag{30}$$

Reducing the order, we can always approximate the values of c_1, by the following expression,

$$c_1^4a_1a_3(b_2-a_7)^4-c_1^3a_1^2a_4(b_2-a_7)^3+c_1^2a_1^3a_5(b_2-a_7)^2-c_1a_1^4a_6(b_2-a_7)+a_1^5a_7=0 \tag{31}$$

The 6th order polynomial is separated as,

$$\left(c_1\lambda^5+c_2\lambda^4+c_3\lambda^3+c_4\lambda^2+c_5\lambda+c_6-1\right)\left[(b_1-a_6)\lambda+(b_2-a_7)\right]=0 \text{ or}$$
$$\left(c_1\lambda^5+c_2\lambda^4+c_3\lambda^3+c_4\lambda^2+c_5\lambda+c_6-1\right)\left[(b_1-a_6)\lambda+(b_2-a_7)\right]\approx 0 \tag{32}$$

This proves lemma 3.

Example 3: Consider the 6th order polynomial equation as follows,

$$12\lambda^6+18\lambda^5+50\lambda^4+22\lambda^3-7\lambda^2+48\lambda-13=0 \tag{33}$$

which, the exact results with equation (29) and $(b_2-a_7)=4$ are,

$$c_1=\begin{pmatrix}1.8012+5.8639i\\1.8012-5.8639i\\3.9182\\-1.1103+2.3373i\\-1.1103-2.3373i\\-0.8\end{pmatrix},\ (b_1-a_6)=\begin{pmatrix}0.5744-1.87i\\0.5744+1.87i\\3.0626\\-1.9899-4.1888i\\-1.9899+4.1888i\\-15.0009\end{pmatrix},\ c_2=\begin{pmatrix}13.0821+1.7544i\\13.0821-1.7544i\\0.7599\\-0.2554+5.2361i\\-0.2554-5.2361i\\-1.4132\end{pmatrix},\ c_3=\begin{pmatrix}3.0798-2.1912i\\3.0798+2.1912i\\15.3334\\-0.6415+11.8758i\\-0.6415-11.8758i\\-3.71\end{pmatrix},$$

$$c_4 = \begin{pmatrix} \text{-2.8299 + 6.0463i} \\ \text{-2.8299 - 6.0463i} \\ \text{-12.8431} \\ \text{6.9794 + 9.1802i} \\ \text{6.9794 - 9.1802i} \\ \text{-2.4559} \end{pmatrix}, \ c_5 = \begin{pmatrix} \text{12.4667 - 1.5194i} \\ \text{12.4667 + 1.5194i} \\ \text{14.4884} \\ \text{10.3832 - 3.4034i} \\ \text{10.3832 + 3.4034i} \\ \text{-0.1882} \end{pmatrix} \text{ and } c_6 = \begin{pmatrix} \text{-2.25} \\ \text{-2.25} \\ \text{-2.25} \\ \text{-2.25} \\ \text{-2.25} \\ \text{-2.25} \end{pmatrix} \quad (34)$$

which all rows give the exact values. The first and fifth rows give,

First row: $\begin{bmatrix} (1.8012 + 5.8639\text{i})\lambda^5 + (13.0821 + 1.7544\text{i})\lambda^4 + (3.0798 - 2.1912\text{i})\lambda^3 \\ +(-2.8299 + 6.0463\text{i})\lambda^2 + (12.4667 - 1.5194\text{i})\lambda - 3.25 \end{bmatrix} * \left[(0.5744 - 1.87\text{i})\lambda + 4\right]$

$= 12\lambda^6 + 18\lambda^5 + 50\lambda^4 + 22\lambda^3 - 7\lambda^2 + 48\lambda - 13 = 0$

Fifth row: $\begin{bmatrix} (-1.1103 - 2.3373\text{i})\lambda^5 + (-0.2554 - 5.2361\text{i})\lambda^4 + (-0.6415 - 11.8758\text{i})\lambda^3 \\ +(6.9794 - 9.1802\text{i})\lambda^2 + (10.3832 + 3.4034\text{i})\lambda - 3.25 \end{bmatrix} * \left[(-1.9899 + 4.1888\text{i})\lambda + 4\right]$

$= 12\lambda^6 + 18\lambda^5 + 50\lambda^4 + 22\lambda^3 - 7\lambda^2 + 48\lambda - 13 = 0$

Applying lemma 3 with $(b_2 - a_7) = -42.17$ into (31) as follows,

$$12\lambda^6 + 18\lambda^5 + 50\lambda^4 + 22\lambda^3 - 7\lambda^2 + 48\lambda - 13 = 0 \quad (35)$$

with the obtained decomposed coefficients as,

$$c_1 = \begin{pmatrix} \text{-0.3617} \\ \text{0.0802 + 0.2357i} \\ \text{0.0802 - 0.2357i} \\ \text{0.076} \end{pmatrix}, \ (b_1 - a_6) = \begin{pmatrix} \text{-33.18} \\ \text{15.53 - 45.63i} \\ \text{15.53 + 45.63i} \\ \text{157.83} \end{pmatrix}, \ c_2 = \begin{pmatrix} \text{-0.0828} \\ \text{-0.0523 + 0.4865i} \\ \text{-0.0523 - 0.4865i} \\ \text{0.1344} \end{pmatrix}, \ c_3 = \begin{pmatrix} \text{-1.4018} \\ \text{-0.0834 + 1.0759i} \\ \text{-0.0834 - 1.0759i} \\ \text{0.3527} \end{pmatrix},$$

$$c_4 = \begin{pmatrix} \text{1.1186} \\ \text{-0.7677 + 0.6663i} \\ \text{-0.7677 - 0.6663i} \\ \text{0.2336} \end{pmatrix}, \ c_5 = \begin{pmatrix} \text{-1.2107} \\ \text{-0.8152 - 0.5855i} \\ \text{-0.8152 + 0.5855i} \\ \text{0.0181} \end{pmatrix} \text{ and } c_6 = \begin{pmatrix} \text{1.0921} \\ \text{1.5761 + 0.1026i} \\ \text{1.5761 - 0.1026i} \\ \text{1.309} \end{pmatrix} \quad (36)$$

The reader may verify that the closest result is depicted in fourth row as follows,

$$\left(0.076\lambda^5 + 0.1344\lambda^4 + 0.3527\lambda^3 + 0.2336\lambda^2 + 0.0181\lambda + 0.309\right)(157.83\lambda - 42.17)$$
$$= 12\lambda^6 + 18\lambda^5 + 50\lambda^4 + 22\lambda^3 - 7\lambda^2 + 48\lambda - 13.0289 \approx 12\lambda^6 + 18\lambda^5 + 50\lambda^4 + 22\lambda^3 - 7\lambda^2 + 48\lambda - 13 = 0$$

It is important to note that the smaller a_{n+1} will produce more accurate approximations for all cases.

3. The Physical Example: Channel Flow with Porous Walls

Important example is also demonstrated in the generating solution of channel flow with porous walls that is governed by the following relation [9],

$$f_{\eta\eta\eta} + Rf_{\eta}^{2} - Rff_{\eta\eta} - P = 0 \tag{37}$$

with f_{η} is velocity in similarity variables and all coefficients are constants, which known solution requires P dictates boundary conditions. On the other hand, the standard treatment for the application of simplest equation is balancing the highest order by expansion as [10,11],

$$f = b_0 + b_1 g$$

which expands (37) as follows,

$$b_1 g_{\eta\eta\eta} - Rb_0 b_1 g_{\eta\eta} - Rb_1^2 g g_{\eta\eta} + Ra_1^2 g^4 + 2Ra_1 a_2 g^3 + R\left(2a_1 a_3 + a_2^2\right) g^2 + 2Ra_2 a_3 g + Ra_3^2 - P = 0 \tag{38}$$

Applying the simplest equation in the form of constant coefficient Riccati equation [12],

$$g_{\eta} = a_1 g^2 + a_2 g + a_3$$
$$g_{\eta\eta} = 2a_1^2 g^3 + 3a_1 a_2 g^2 + \left(2a_1 a_3 + a_2^2\right) g + a_2 a_3$$
$$g_{\eta\eta\eta} = 6a_1^3 g^4 + 12a_1^2 a_2 g^3 + \left(8a_1^2 a_3 + 7a_1 a_2^2\right) g^2 + \left(8a_1 a_2 a_3 + a_2^3\right) g + 2a_1 a_3^2 + a_2^2 a_3 + a_2 a_3$$
$$g_{\eta}^2 = a_1^2 g^4 + 2a_1 a_2 g^3 + \left(2a_1 a_3 + a_2^2\right) g^2 + 2a_2 a_3 g + a_3^2$$

whose solution appears to be [13,14],

$$g_{\eta} = a_1\left(g^2 + \frac{a_2}{a_1} g + \frac{a_3}{a_1}\right) \text{ or } g_{\eta} = \left(a_1 g + \frac{a_2}{2} + \frac{1}{2}\sqrt{a_2^2 - 4a_1 a_3}\right)\left(g + \frac{a_2}{2a_1} - \frac{1}{2}\sqrt{\left(\frac{a_2}{a_1}\right)^2 - \frac{4a_3}{a_1}}\right) \text{ or}$$

$$\frac{1}{\left(a_1 g + \frac{a_2}{2} + \frac{1}{2}\sqrt{a_2^2 - 4a_1 a_3}\right)\left(g + \frac{a_2}{2a_1} - \frac{1}{2}\sqrt{\left(\frac{a_2}{a_1}\right)^2 - \frac{4a_3}{a_1}}\right)} dg = d\eta \text{ or}$$

$$\frac{\left(a_2^2 - 4a_1 a_3\right)^{-\frac{1}{2}}}{\left(g + \frac{a_2}{2a_1} - \frac{1}{2}\sqrt{\left(\frac{a_2}{a_1}\right)^2 - \frac{4a_3}{a_1}}\right)} dg - \frac{\left(a_2^2 - 4a_1 a_3\right)^{-\frac{1}{2}}}{\left(g + \frac{a_2}{2a_1} + \frac{1}{2}\sqrt{\left(\frac{a_2}{a_1}\right)^2 - \frac{4a_3}{a_1}}\right)} dg = d\eta \text{ or}$$

$$\ln\left(g + \frac{a_2}{2a_1} - \frac{1}{2}\sqrt{\left(\frac{a_2}{a_1}\right)^2 - \frac{4a_3}{a_1}}\right) - \ln\left(g + \frac{a_2}{2a_1} + \frac{1}{2}\sqrt{\left(\frac{a_2}{a_1}\right)^2 - \frac{4a_3}{a_1}}\right) = \left(a_2^2 - 4a_1 a_3\right)^{\frac{1}{2}} \eta + \gamma \text{ or}$$

$$g = \frac{1}{2}\sqrt{\left(\frac{a_2}{a_1}\right)^2 - \frac{4a_3}{a_1}}\left(\frac{1 + \gamma e^{\sqrt{a_2^2 - 4a_1 a_3}\,\eta}}{1 - \gamma e^{\sqrt{a_2^2 - 4a_1 a_3}\,\eta}}\right) - \frac{a_2}{2a_1} \tag{39}$$

Here, the reader should not be confused with the repeating symbols in decomposition and Riccati coefficients as they are in different role and context. Substituting the Riccati equation into (38) as,

$$
\begin{aligned}
& b_1 g_{\eta\eta\eta} - Rb_0 b_1 g_{\eta\eta} - Rb_1^2 g g_{\eta\eta} + Ra_1^2 g^4 + 2Ra_1 a_2 g^3 + R\left(2a_1 a_3 + a_2^2\right) g^2 + 2Ra_2 a_3 g + Ra_3^2 - P = 0 \\
& 6a_1^3 b_1 g^4 + 12a_1^2 a_2 b_1 g^3 + \left(8a_1^2 a_3 + 7a_1 a_2^2\right) b_1 g^2 + \left(8a_1 a_2 a_3 + a_2^3\right) b_1 g + 2a_1 a_3^2 b_1 + a_2^2 a_3 b_1 + a_2 a_3 b_1 \\
& -2a_1^2 Rb_0 b_1 g^3 - 3a_1 a_2 Rb_0 b_1 g^2 - \left(2a_1 a_3 + a_2^2\right) Rb_0 b_1 g - a_2 a_3 Rb_0 b_1 \qquad \text{or} \\
& -2a_1^2 Rb_1^2 g^4 - 3a_1 a_2 Rb_1^2 g^3 - \left(2a_1 a_3 + a_2^2\right) Rb_1^2 g^2 - a_2 a_3 Rb_1^2 g \\
& + Ra_1^2 g^4 + 2Ra_1 a_2 g^3 + R\left(2a_1 a_3 + a_2^2\right) g^2 + 2Ra_2 a_3 g + Ra_3^2 - P = 0 \\
& \left(6a_1^3 b_1 - 2a_1^2 Rb_1^2 + Ra_1^2\right) g^4 + \left(12a_1^2 a_2 b_1 + 2Ra_1 a_2 - 2a_1^2 Rb_0 b_1 - 3a_1 a_2 Rb_1^2\right) g^3 \\
& + \left(8a_1^2 a_3 b_1 + 7a_1 a_2^2 b_1 - 3a_1 a_2 Rb_0 b_1 - 2a_1 a_3 Rb_1^2 - a_2^2 Rb_1^2 + 2Ra_1 a_3 + Ra_2^2\right) g^2 \\
& + \left(8a_1 a_2 a_3 b_1 + 2Ra_2 a_3 + a_2^3 b_1 - 2a_1 a_3 Rb_0 b_1 - a_2^2 Rb_0 b_1 - a_2 a_3 Rb_1^2\right) g \\
& + 2a_1 a_3^2 b_1 + a_2^2 a_3 b_1 + a_2 a_3 b_1 - a_2 a_3 Rb_0 b_1 + Ra_3^2 - P = 0
\end{aligned} \tag{40}
$$

The coefficient relations give,

$$
\begin{aligned}
& 6a_1^3 b_1 - 2a_1^2 Rb_1^2 + Ra_1^2 = 0 \\
& 12a_1^2 a_2 b_1 + 2Ra_1 a_2 - 2a_1^2 Rb_0 b_1 - 3a_1 a_2 Rb_1^2 = 0 \\
& 8a_1^2 a_3 b_1 + 7a_1 a_2^2 b_1 - 3a_1 a_2 Rb_0 b_1 - 2a_1 a_3 Rb_1^2 - a_2^2 Rb_1^2 + 2Ra_1 a_3 + Ra_2^2 = 0 \quad \text{or} \\
& 8a_1 a_2 a_3 b_1 + a_2^3 b_1 - 2a_1 a_3 Rb_0 b_1 + 2Ra_2 a_3 - a_2^2 Rb_0 b_1 - a_2 a_3 Rb_1^2 = 0 \\
& 2a_1 a_3^2 b_1 + a_2^2 a_3 b_1 + a_2 a_3 b_1 - a_2 a_3 Rb_0 b_1 + Ra_3^2 - P = 0 \\
& 6a_1 b_1 = 2Rb_1^2 - R \\
& a_2 b_1^2 = 4Rb_0 b_1^2 - 2Rb_0 = 2b_0\left(2Rb_1^2 - R\right) \\
& a_3 b_1^5 + a_3 b_1^3 = 59Rb_0^2 b_1^2 - 86Rb_0^2 b_1^4 - 8Rb_0^2 = Rb_0^2\left(59b_1^2 - 86b_1^4 - 8\right) \\
& \left(9b_1^2 + 4\right)\left(59b_1^2 - 86b_1^4 - 8\right) + \left(72b_1^4 - 84b_1^2 + 24\right)\left(b_1^2 + 1\right) = 0 \\
& b_0^4 b_1^3\left(2Rb_1^2 + 2R\right)\left(\frac{59Rb_1^2 - 86Rb_1^4 - 8R}{b_1^5 + b_1^3}\right)^2 + 12b_0^3\left(2Rb_1^2 - R\right)^2\left(\frac{59Rb_1^2 - 86Rb_1^4 - 8R}{b_1^5 + b_1^3}\right) \\
& + 6b_0^2 b_1^2\left(2Rb_1^2 - R\right)\left(\frac{59Rb_1^2 - 86Rb_1^4 - 8R}{b_1^5 + b_1^3}\right) - 6Rb_0^3 b_1^2\left(2Rb_1^2 - R\right)\left(\frac{59Rb_1^2 - 86Rb_1^4 - 8R}{b_1^5 + b_1^3}\right) - 3Pb_1^3 = 0
\end{aligned} \tag{41}
$$

which solve all unknown coefficients, a_i and b_j. Unlike the condition in previous solution [9], P in (37) may be determined a priori. Therefore, the solution of porous channel flow will be,

$$
f = b_0 + \frac{b_1}{2}\sqrt{\left(\frac{a_2}{a_1}\right)^2 - \frac{4a_3}{a_1}}\left(\frac{1 + \gamma e^{\sqrt{a_2^2 - 4a_1 a_3}\,\eta}}{1 - \gamma e^{\sqrt{a_2^2 - 4a_1 a_3}\,\eta}}\right) - \frac{b_1 a_2}{2a_1} \tag{42}
$$

Meanwhile, direct substitution of Riccati equation into (37) without balancing order is,

$$f_{\eta\eta\eta} + Rf_{\eta}^2 - Rff_{\eta\eta} - P = 0$$

$$6a_1^3 f^4 + 12a_1^2 a_2 f^3 + \left(8a_1^2 a_3 + 7a_1 a_2^2\right) f^2 + \left(8a_1 a_2 a_3 + a_2^3\right) f + 2a_1 a_3^2 + a_2^2 a_3 + a_2 a_3$$
$$+Ra_1^2 f^4 + 2Ra_1 a_2 f^3 + R\left(2a_1 a_3 + a_2^2\right) f^2 + 2Ra_2 a_3 f + Ra_3^2 \qquad \text{or}$$
$$-2Ra_1^2 f^4 + 3Ra_1 a_2 f^3 + R\left(2a_1 a_3 + a_2^2\right) f^2 + Ra_2 a_3 f - P = 0$$

$$\left(6a_1^3 - Ra_1^2\right) f^4 + \left(12a_1^2 a_2 + 5Ra_1 a_2\right) f^3 + \left(4Ra_1 a_3 + 2Ra_2^2 + 8a_1^2 a_3 + 7a_1 a_2^2\right) f^2 + \left(8a_1 a_2 a_3 + a_2^3 + 3Ra_2 a_3\right) f \qquad \text{or}$$
$$+2a_1 a_3^2 + a_2^2 a_3 + a_2 a_3 + Ra_3^2 - P = 0$$

$$A_1 f^4 + A_2 f^3 + A_3 f^2 + A_5 f + A_5 = 0 \tag{43}$$

which have more polynomial coefficients, A_n than unknown Riccati coefficients, a_n. Application of Lemma 1 dictates the decomposition expressions are,

$$\left(c_1 f^3 + c_2 f^2 + c_3 f + c_4 - 1\right)\left[\left(b_1 - A_4\right) f + \left(b_2 - A_5\right)\right] = 0$$

for exact decomposition or,

$$\left(c_1 f^3 + c_2 f^2 + c_3 f + c_4 - 1\right)\left[\left(b_1 - A_4\right) f + \left(b_2 - A_5\right)\right] \approx 0$$

for approximation using order reduction. Since the initial coefficients $\left(b_2 - A_5\right)$ can be chosen arbitrarily, the zero or near-zero conditions can be satisfied by,

$$\left(b_1 - A_4\right) = \frac{6a_1^3 - Ra_1^2}{c_1} = 0 \quad \text{or} \quad 6a_1 - R = 0$$
$$\left(b_2 - A_5\right) = 2a_1 a_3^2 + a_2^2 a_3 + a_2 a_3 + Ra_3^2 - P = 0 \tag{44}$$

Eq. (44) thus determine a_1, a_2 and a_3, the solution for the porous channel flow is,

$$f = \frac{1}{2}\sqrt{\left(\frac{a_2}{a_1}\right)^2 - \frac{4a_3}{a_1}}\left(\frac{1 + \gamma e^{\sqrt{a_2^2 - 4a_1 a_3}\,\eta}}{1 - \gamma e^{\sqrt{a_2^2 - 4a_1 a_3}\,\eta}}\right) - \frac{a_2}{2a_1} \tag{45}$$

Note that one may generalize the solution by setting (45) as a particular solution, f_0 and the general solution would be [15],

$$f = f_0 + e_1 f_1 + e_2 f_2 + \ldots\ldots\ldots$$

This generalization by superposition or by product superposition will extend the problem into the solution of (43) with variable coefficients Riccati equation and would be the subject of future work by implementing the proposed polynomial decomposition.

4. Conclusions

By induction, lemma 1, lemma 2 and lemma 3 lead to the more general statement as in the following theorem,

Theorem. Let, h and β are any non-zero arbitrary functions that decompose the higher order polynomial,

$$a_1\lambda^n + a_2\lambda^{n-1} + a_3\lambda^{n-2} + a_4\lambda^{n-3} + a_5\lambda^{n-4} + ... + a_{n-1}\lambda^2 + a_n\lambda + a_{n+1} = 0$$

The polynomial can then be factorized as,

$$\left(c_1\lambda^{n-1} + c_2\lambda^{n-2} + c_3\lambda^{n-3} + c_4\lambda^{n-4} + c_5\lambda^{n-5} .... + c_n - 1\right)\left[\left(b_1 - a_n\right)\lambda + \left(b_2 - a_{n+1}\right)\right] = 0 \text{ or}$$
$$\left(c_1\lambda^{n-1} + c_2\lambda^{n-2} + c_3\lambda^{n-3} + c_4\lambda^{n-4} + c_5\lambda^{n-5} .... + c_n - 1\right)\left[\left(b_1 - a_n\right)\lambda + \left(b_2 - a_{n+1}\right)\right] \approx 0$$

with,

$$c_1\left(b_1 - a_n\right) = a_1$$
$$c_2a_1 = -c_1^2\left(b_2 - a_{n+1}\right) + c_1a_2$$
$$c_3a_1 = -c_1c_2\left(b_2 - a_{n+1}\right) - c_1^2\left(b_3 - a_{n+2}\right) + c_1a_3$$
$$c_4a_1 = -c_1c_3\left(b_2 - a_{n+1}\right) - c_1c_2\left(b_3 - a_{n+2}\right) + c_1a_4$$
.
.
$$c_na_1 = -c_1c_{n-2}\left(b_2 - a_{n+1}\right) - c_1c_{n-3}\left(b_3 - a_{n+2}\right) + c_1b_1$$
$$c_n\left(b_2 - a_{n+1}\right) + c_{n-1}\left(b_3 - a_{n+2}\right) = b_2$$
$$c_n\left(b_3 - a_{n+2}\right) = \left(b_3 - a_{n+2}\right)$$

Considering the theorem, experience reader may point out that the exact results are also algebraically fulfilled with, $c_6 = 1$ or $c_n = 1$ in the theorem. However, the numerical trials depict that we only have one exact result after tuning the ratio of $\frac{\left(b_1 - a_5\right)}{\left(b_2 - a_6\right)}$ into a specific value for quintic equation. Thus, using $c_5 = 1$ is not practical and hardly be useful for implementation in the application of simplest equations to non-linear differential equations.

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