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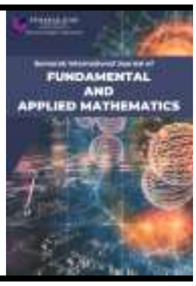

The Union Prime Power Order Cayley Graph of Certain Cyclic Groups and Their Topological Indices

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ABSTRACT

Cayley graph is a variation of graphs that focuses on constructing and analyzing graphs using algebraic structures. Meanwhile, topological indices of graphs are numerical values that reflect various aspects of the graphs' structure. Over the years, many variants of Cayley graphs have been constructed due to the significance of understanding the order of elements within a group's subset but not on the union of subsets with specific order of elements. In this paper, a new variant of Cayley graph, namely the union prime power order Cayley graph of a group G with respect to subset H is formed by combining all possible subsets of G. In addition, the union prime power order Cayley graph is constructed for cyclic groups of order p^2 and p^3, where p is a prime, and their generalizations are determined. Moreover, the topological indices, which are the Wiener index, mean distance, first Zagreb index, and second Zagreb index are also computed for these graphs. Existing definitions and theorems of Cayley graphs and topological indices are analyzed to define the new variant of the Cayley graph and establish the general form of its topological indices.

1. Introduction

In recent years, graph theory has been widely applied in fields such as computer science, social networks, robotics, neuroscience, and many more. Graph theory is based on graphs, which consist of vertices, connected by edges that represent the relationships between the vertices. Algebraic graph theory, on the other hand, involves the study of graphs using algebraic structures such as groups. In 1878, Cayley [1] introduced Cayley graph, a type of graph that represents the abstract algebraic structure of a group based on a set of generators. A Cayley graph of a finite group G with respect to a subset S, denoted as $Cay(G, S)$, has the elements of G as vertices, with the adjacencies of the vertices depending on the subset S of G.

The study of Cayley graphs related to group theory has advanced significantly over time. Konstantinova [2] explored the use of various types of Cayley graphs in solving combinatorial, graph-theoretical, and applied problems in fields such as mathematics, computer science, and biology. In

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2015, Tolue [3] introduced a new variant of Cayley graph, called the prime order Cayley graph associated to a group G and a subset S consisting of prime order elements of G. Asrari and Tolue [4] further investigated the prime order Cayley graph introduced in [3] for certain groups, including cyclic, dihedral, and generalized quaternion groups. Additionally, Tolue [5] defined the composite order Cayley graph of a group G and a subset S of all composite order elements of G, which is the complement of the prime order Cayley graph from [3]. Recently, Zulkarnain *et al*., [6] presented another variant of Cayley graph, called the prime power Cayley graph of a group G, with the subset S of G containing all elements with prime power order. Furthermore, several other variants of Cayley graphs have been explored (see [7,8]). Moreover, the applications of Cayley graphs have expanded beyond mathematics. For example, in [9], Cayley graphs of semigroups are used to track atoms in complex biochemical networks and predict their potential locations during chemical reactions in molecules. Thilaga and Sarasija [10] investigated the use of unitary Cayley graphs in small-world networks, focusing on optimizing network communication in terms of delay and reliability.

In mathematical chemistry, topological indices hold historical significance in predicting the biological activity and physicochemical properties of chemical compounds using molecular graphs. These indices are numerical values derived from mathematical formulas that quantify various aspects of a graph's structure [11,12]. Among the most widely used topological indices are the Wiener index [13] and the Zagreb indices [14]. In [15], the mean distance of a graph is calculated using the Wiener index. These indices have also become more prevalent in algebraic graph theory, particularly in the study of Cayley graphs. In [16], Cayley graphs, particularly circulant and cube-connected graphs, and graph invariants such as forwarding and optical indices, bisection width, and Wiener index are analyzed to enhance efficient network communication. Yancheshmeh *et al*., [17] constructed the Cayley graph of the dihedral and generalized quaternion groups on specific subsets, and computed the Wiener index, Szeged index, and PI index of these graphs. Shojaee *et al*., [18] presented some new results for the Cayley graphs in [3] and [5] of certain abelian groups, focusing on various topological indices, such as the Wiener index, first and second Zagreb indices, eccentric connectivity index, and vertex and edge Padmakar-Ivan indices. Alhubairah *et al*., [19] introduced a new graph, the p-Subgroup graph, and computed the Wiener index, first and second Zagreb indices for these graphs of dihedral group of order $2n$, where $n = p^r$, $r \in \mathbb{N}$ and p is a prime.

Most research on Cayley graphs has focused on finite groups with individual subsets containing elements of prime order [2], composite order [5], and prime power order [6]. However, the union of subsets with specific order of elements of a group has not been explored for Cayley graphs. In this paper, we introduce a new variant of Cayley graph, called the union prime power order Cayley graph for a finite group G with respect to subset H formed by combining all possible subsets of G. This graph is constructed for cyclic groups of order p^2 and p^3, where p is a prime. In addition, several topological indices which are the Wiener index, mean distance, and first and second Zagreb indices are computed, providing new insights into the structure of these graphs in algebraic graph theory.

In the following sections, we provide the basic concepts of graph theory and group theory, and previous results related to various topological indices of graphs. Next, the general structure of the new variant of Cayley graph and the computation of its topological indices are discussed. Finally, the paper concludes with a summary of the research findings.

2. Preliminaries

2.1 Graph

Graphs are mathematical structures used to represent relationships between objects or entities, first introduced by Euler in 1736 while solving the notable Königsberg bridges problem [20].

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a graph, where $V(\Gamma)$ is the set of vertices and $E(\Gamma)$ is the set of edges. In a graph Γ, two vertices u and v are adjacent, denoted $u \sim v$, if there is an edge $\{u, v\}$ connecting them. The order of Γ is the number of vertices, denoted by $|V(\Gamma)|$, while the size of Γ is the number of edges, denoted by $|E(\Gamma)|$. For a vertex $u \in V(\Gamma)$, the degree of u, denoted by $deg(u)$ is the number of vertices adjacent to u. A graph Γ with n vertices is called a complete graph, denoted by K_n, if every distinct pair of vertices in Γ is adjacent. A graph Γ is d-regular if every vertex in Γ has the same degree d, and the size of the regular graph Γ with n vertices is $\frac{nd}{2}$. Furthermore, a graph Γ is considered connected if there exists a path between every pair of vertices, otherwise, Γ is a disconnected graph. Let $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_k$ be the union of k graphs, each with disjoint vertex and edge sets, where $V(\Gamma) = V(\Gamma_1) \cup V(\Gamma_2) \cup \cdots \cup V(\Gamma_k)$, and $E(\Gamma) = E(\Gamma_1) \cup E(\Gamma_2) \cup \cdots \cup E(\Gamma_k)$. The union of k disjoint copies of Γ is denoted by $k\Gamma$. The distance between two vertices u and v in a connected graph Γ, denoted by $d(u, v)$, is the length of the shortest path between them, while the diameter of Γ, denoted $diam(\Gamma)$, is the maximum distance between any two vertices in Γ [21].

According to [22], a group G is called cyclic if there exists an element $a \in G$ such that G is generated by a.

In group theory, graphs are used to represent the algebraic structure of a group. Cayley graphs are a type of graph of groups, where vertices correspond to the elements of the group, and edges represent the group's action, providing a graphical representation of the group's structure.

Definition 1 [1] Cayley Graph

Let G be a group and S a subset of G such that the identity element $e \notin S$ and $S = S^{-1}$. The Cayley graph of G with respect to S, denoted $Cay(G, S)$, has vertex set $V(Cay(G, S)) = G$, and two vertices g and h are adjacent if and only if $gh^{-1} \in S$.

In 2022, Zulkarnain *et al.*, [6] introduced the prime power Cayley graph of a group G and a subset S of G containing all prime power order elements. For a cyclic group G of prime power order, it was shown that the prime power Cayley graph forms a complete graph.

Theorem 1 [6]

Let G be a cyclic group of order p^n, where p is a prime and $n \in \mathbb{N}$. Let S be a subset of G defined by $S = \{x \in G: |x| = p^k, 1 \leq k \leq n\}$ and $S = S^{-1}$. Then, the prime power Cayley graph of G with respect to S, denoted $\widetilde{Cay}(G, S)$, is a complete graph of order p^n, that is $\widetilde{Cay}(G, S) = K_{p^n}$.

In this paper, by extending the idea in [6], a new variant of Cayley graph is introduced, namely the union prime power order Cayley graph of a group G with respect to the union of all possible subsets of G.

2.2 Topological Indices of Graphs

Topological indices are numerical values that represent the topological structure of a graph. Various topological indices have been introduced over the past decades, including the Wiener index, mean distance, and Zagreb indices. Here, the definitions, theorems, and propositions related to the topological indices of graphs are given.

Definition 2 [13] Wiener Index of a Graph

The Wiener index of a connected graph Γ is, $W(\Gamma) = \sum_{\{u,v\} \in V(\Gamma)} d(u, v)$, where $d(u, v)$ is the distance between two vertices u and v in Γ.

Proposition 1 [23]
For a complete graph with n vertices, K_n, the Wiener index is, $W(K_n) = \frac{n(n-1)}{2}$.

Theorem 1 [24]
Let Γ be a graph with $|V(\Gamma)|$ vertices and $|E(\Gamma)|$ edges. Then, the Wiener index is, $W(\Gamma) = |V(\Gamma)|^2 - |V(\Gamma)| - |E(\Gamma)|$ if and only if $diam(\Gamma) \leq 2$.

Proposition 2 [23]
For a disconnected graph Γ, the Wiener index is, $W(\Gamma) = \sum_{\substack{\{u,v\}\subseteq V(\Gamma) \\ u-v \text{ path exists in } \Gamma}} d\,(u, v)$, where $d(u, v)$ is the distances between the vertices u and v within each connected component of Γ.

Definition 3 [15] Mean Distance of a Graph
The mean distance of a graph Γ is, $\sigma(\Gamma) = \frac{W(\Gamma)}{\binom{|V(\Gamma)|}{2}}$, where $W(\Gamma)$ is the Wiener index of graph Γ and $|V(\Gamma)|$ is the number of vertices in Γ.

Proposition 3 [15]
For a complete graph with n vertices, K_n, the mean distance is, $\sigma(K_n) = 1$.

Definition 4 [14] First Zagreb Index of a Graph
For a connected graph Γ, the first Zagreb index is, $M_1(\Gamma) = \sum_{v\in V(\Gamma)} deg(v)^2$, where $deg(v)$ is the degree of vertex v in Γ.

Definition 5 [14] Second Zagreb Index of a Graph
For a connected graph Γ, the second Zagreb index is, $M_2(\Gamma) = \sum_{\{u,v\}\in E(\Gamma)} deg(u)deg(v)$, where $deg(u)$ and $deg(v)$ represent the degrees of adjacent vertices u and v in Γ.

Proposition 4 [25]
For a complete graph with n vertices, K_n, the first Zagreb index is, $M_1(K_n) = n(n-1)^2$.

Proposition 5 [25]
For a complete graph with n vertices, K_n, the second Zagreb index is, $M_2(K_n) = \frac{n(n-1)^3}{2}$.

Proposition 6 [25]
Let Γ be a d-regular graph with n vertices, then the first Zagreb index is, $M_1(\Gamma) = nd^2$, where d is the degree of each vertex in Γ.

Proposition 7 [25]
Let Γ be a d-regular graph with n vertices, then the second Zagreb index is, $M_2(\Gamma) = \frac{nd^3}{2}$, where d is the degree of each vertex in Γ.

Proposition 8 [26]
Let Γ be a disconnected graph with components $\Gamma_1, \Gamma_2, \dots, \Gamma_k$, then the first Zagreb index is, $M_1(\Gamma) = M_1(\Gamma_1) + M_1(\Gamma_2) + \cdots + M_1(\Gamma_k)$.

Proposition 9 [26]
Let Γ be a disconnected graph with components $\Gamma_1, \Gamma_2, \dots, \Gamma_k$, then the second Zagreb index is, $M_2(\Gamma) = M_2(\Gamma_1) + M_2(\Gamma_2) + \cdots + M_2(\Gamma_k)$.

In the next section, we introduce a new variant of Cayley graph, called the union prime power order Cayley graph. Then, the structures of these graphs for cyclic groups of order p^2 and p^3, where p is a prime, and their topological indices, are determined.

3. Results

In this research, a new variant of Cayley graph, namely the union prime power order Cayley graph is introduced. The union prime power order Cayley graph is defined as follows:

Definition 6 Union Prime Power Order Cayley Graph
Let G be a group with $|G| = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \dots \cdot p_k^{\alpha_k}$ where p_i are primes and $\alpha_i \in \mathbb{N}$ for $i = 1,2,\dots,k$. Suppose $S^{(p_i^r)} = \{a \in G: |a| = p_i^r\}$ be an inverse-closed subset of G for $r = 1,2,\dots,\alpha_i$ and H be a subset formed from all possible union of $S^{(p_i^r)}$. The union prime power order Cayley graph of G with respect to H, denoted as $UCay_{pp}(G,H)$ is a graph with the elements of G as its vertices and two distinct vertices g and h are adjacent if $gh^{-1} \in H$.

Example 1
Let G be a cyclic group of order 8, $|C_8| = 8 = 2^3$, generated by x with the set of elements $\{e, x, x^2, x^3, x^4, x^5, x^6, x^7\}$. By 0, there are three subsets associated to G, which are:

i) $S^{(2)} = \{x^r \in G: |x^r| = 2\} = \{x^4\}$,
ii) $S^{(4)} = \{x^r \in G: |x^r| = 4\} = \{x^2, x^6\}$,
iii) $S^{(8)} = \{x^r \in G: |x^r| = 8\} = \{x, x^3, x^5, x^7\}$.

Hence, the possible unions of subsets $S^{(2)}$, $S^{(4)}$ and $S^{(3)}$ can be formed as follows:

i) $H_1 = S^{(2)} \cup S^{(4)} = \{x^r \in G: |x^r| = 2 \text{ or } 4\} = \{x^2, x^4, x^6\}$,
ii) $H_2 = S^{(2)} \cup S^{(8)} = \{x^r \in G: |x^r| = 2 \text{ or } 8\} = \{x, x^3, x^4, x^5, x^7\}$,
iii) $H_3 = S^{(4)} \cup S^{(8)} = \{x^r \in G: |x^r| = 4 \text{ or } 8\} = \{x, x^2, x^3, x^5, x^6, x^7\}$,
iv) $H_4 = S^{(2)} \cup S^{(4)} \cup S^{(8)} = \{x^r \in G: |x^r| = 2, 4 \text{ or } 8\} = \{x, x^2, x^3, x^4, x^5, x^6, x^7\}$.

Based on 0, all the graphs have elements of G as vertices, and the vertices of the graphs are adjacent if $gh^{-1} \in H_1, H_2, H_3, H_4$. Therefore, the union prime power order Cayley graph of G with respect to the subsets H_1, H_2, H_3, and H_4, are obtained, as in Figures 1-4, respectively.

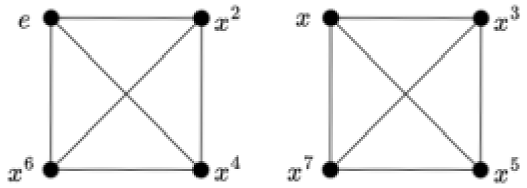

Fig. 1. $UCay_{pp}(G, H_1)$

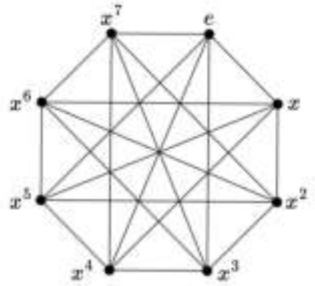

Fig. 2. $UCay_{pp}(G, H_2)$

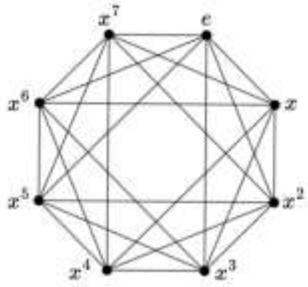

Fig. 3. $UCay_{pp}(G, H_3)$

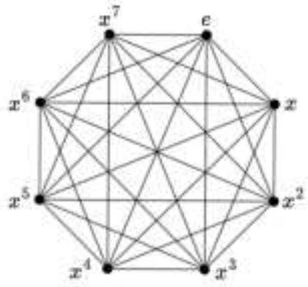

Fig. 4. $UCay_{pp}(G, H_4)$

Note that the union prime power order Cayley graph cannot be constructed for a cyclic group G of order p, where p is a prime since by 0, there is only one subset associated to G, which is $S^{(p)} = \{x^r \in G: |x^r| = p\}$. Thus, it is not possible to get a union of subsets of elements with distinct prime power order.

Therefore, we start by constructing the union prime power order Cayley graph for the cyclic groups of order p^2, where p is a prime. Then, we determine the Wiener index, mean distance, first Zagreb index, and second Zagreb index for this graph.

3.1 The Union Prime Power Order Cayley Graph of Cyclic Groups of Order p^2

Let G be a cyclic group of order p^2, for prime p. For any element $x^r \in G$, the order of x^r can be 1, p, or p^2. By 0, there are two subsets associated to G, which are:

i) $S^{(p)} = \{x^r \in G: |x^r| = p\}$,
ii) $S^{(p^2)} = \{x^r \in G: |x^r| = p^2\}$.

Thus, a subset H can be formed by the union of $S^{(p)}$ and $S^{(p^2)}$, as follows:

$$H = S^{(p)} \cup S^{(p^2)} = \{x^r \in G: |x^r| = p \text{ or } p^2\}.$$

The union prime power order Cayley graph of G with respect to H is constructed as in the following theorem.

Theorem 3

Let G be a cyclic group of order p^2 generated by x, where p is a prime. Let H be a subset of G defined as $H = S^{(p)} \cup S^{(p^2)} = \{x^r \in G: |x^r| = p \text{ or } p^2\}$. The union prime power order Cayley graph of G with respect to H is a complete graph of order p^2, that is $UCay_{pp}(G, H) = K_{p^2}$.

Proof. Let G be a cyclic group of order p^2 generated by x, where $G = \{e, x, \dots, x^p, x^{p+1}, \dots, x^{p^2-1}\}$.

Let H be a subset of G in which $H = S^{(p)} \cup S^{(p^2)} = \{x^r \in G: |x^r| = p \text{ or } p^2\}$, includes all elements in G except the identity element. Thus, $H = \{x^i \in G: 1 \leq i \leq (p^2 - 1)\}$.

By 0, $UCay_{pp}(G, H)$ has vertex set $V(UCay_{pp}(G, H)) = G$. For $1 \leq i \leq (p^2 - 1)$, each vertex x^i is adjacent to the identity element e, since $x^i \cdot e^{-1} = x^i \in H$. Two distinct vertices, x^i and x^j in $V((UCay_{pp}(G, H))$ are adjacent since $x^i \cdot x^{-j} \neq e$, as $x^i \cdot x^{-j} = e$ implies $x^i = x^j$, which is a contradiction since x^i and x^j are distinct. Therefore, $x^i \cdot x^{-j} \in H$ and $x^i \sim x^j$.

Since every pair of distinct vertices in $UCay_{pp}(G, H)$ are adjacent, the graph forms a complete graph of order p^2, denoted K_{p^2}. Therefore, $UCay_{pp}(G, H) = K_{p^2}$.

Example 2

Let G be a cyclic group of order 4, C_4, generated by x with the set of elements $\{e, x, x^2, x^3\}$. By 0, there are two subsets associated to G, which are:

i) $S^{(2)} = \{x^r \in G: |x^r| = 2\} = \{x^2\}$,
ii) $S^{(4)} = \{x^r \in G: |x^r| = 4\} = \{x, x^3\}$.

Hence, a subset H can be formed by the union of $S^{(2)}$ and $S^{(4)}$, as follows:

$H = S^{(2)} \cup S^{(4)} = \{x^r \in G: |x^r| = 2 \text{ or } 4\} = \{x, x^2, x^3\}$.

The union prime power order Cayley graph of G with respect to H, $UCay_{pp}(G, H)$ has vertex set $V(UCay_{pp}(G, H)) = G = \{e, x, x^2, x^3\}$. The vertex e is adjacent to all other vertices in $UCay_{pp}(G, H)$ since $x^i \cdot e^{-1} = x^i \in H$ for $i = 1,2,3$. Similarly, $x^i \sim x^j$ for all $i \neq j, 1 \leq i, j \leq 3$ since all elements of G are in H except for the identity.

Therefore, $UCay_{pp}(G, H)$ is a complete graph of order 4, as shown in Figure 5.

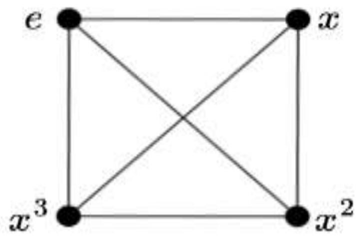

Fig. 5. $UCay_{pp}(G, H)$

3.2 The Topological Indices of the Union Prime Power Order Cayley Graph of Cyclic Groups of Order p^2

For a cyclic group G of order p^2, where p is a prime, the union prime power order Cayley graph of G with respect to the subset $H = S^{(p)} \cup S^{(p^2)} = \{x^r \in G: |x^r| = p \text{ or } p^2\}$ is a complete graph with p^2 vertices, K_{p^2}, as shown in 0.

In the following theorem, the Wiener index, mean distance, first Zagreb index, and second Zagreb index of the union prime power order Cayley graphs of a cyclic group G of order p^2, for prime p, with respect to the subset H, denoted by $W(UCay_{pp}(G,H))$, $\sigma(UCay_{pp}(G,H))$, $M_1(UCay_{pp}(G,H))$ and $M_2(UCay_{pp}(G,H))$ are determined.

Theorem 4

Let G be a cyclic group of order p^2 and $H = S^{(p)} \cup S^{(p^2)} = \{x^r \in G: |x^r| = p \text{ or } p^2\}$. Then,

i) $W(UCay_{pp}(G,H)) = \frac{p^2(p^2-1)}{2}$,
ii) $\sigma(UCay_{pp}(G,H)) = 1$,
iii) $M_1(UCay_{pp}(G,H)) = p^2(p^2-1)^2$,
iv) $M_2(UCay_{pp}(G,H)) = \frac{p^2(p^2-1)^3}{2}$.

Proof. From 0, $UCay_{pp}(G,H) = K_{p^2}$. Let $n = \left|V(UCay_{pp}(G,H))\right|$. By using 0, the Wiener index of $UCay_{pp}(G,H)$ is,

$$W(UCay_{pp}(G,H)) = W(K_{p^2}) = \frac{p^2(p^2-1)}{2}.$$

Based on 0, the mean distance of any complete graph is one. Hence, $\sigma(UCay_{pp}(G,H)) = \sigma(K_{p^2}) = 1$.

Next, the first and second Zagreb indices are determined using 0 and 0, respectively:

$$M_1\left(UCay_{pp}(G,H)\right) = M_1(K_{p^2}) = p^2(p^2-1)^2, M_2(UCay_{pp}(G,H)) = M_2(K_{p^2}) = \frac{p^2(p^2-1)^3}{2}.$$

Example 3

From 0, for a cyclic group $G = C_4$ and subset $H = S^{(2)} \cup S^{(4)} = \{x^r \in G: |x^r| = 2 \text{ or } 4\} = \{x, x^2, x^3\}$, $UCay_{pp}(G,H) = K_4$. Hence, the distance between distinct vertices is 1.

By 0,

$$\begin{aligned} W(UCay_{pp}(G,H)) &= \sum_{\{u,v\} \in V(UCay_{pp}(G,H))} d(u,v) \\ &= d(e,x) + d(e,x^2) + d(e,x^3) + d(x,x^2) + d(x,x^3) + d(x^2,x^3) \\ &= 6, \end{aligned}$$

and by 0,

$$\sigma(UCay_{pp}(G,H)) = \frac{W(UCay_{pp}(G,H))}{\binom{|V(UCay_{pp}(G,H))|}{2}} = \frac{6}{\binom{4}{2}} = 1.$$

Next, for every vertex $v \in V(UCay_{pp}(G,H))$, the degree, $deg(v) = 3$. Therefore, by 0 and 0, respectively,

$$\begin{aligned} M_1(UCay_{pp}(G,H)) &= \sum_{v \in V(UCay_{pp}(G,H))} deg(v)^2 \\ &= deg(e)^2 + deg(x)^2 + deg(x^2)^2 + deg(x^3)^2 \\ &= 36, \end{aligned}$$

$$\begin{aligned} M_2(UCay_{pp}(G,H)) &= \sum_{\{u,v\} \in E(UCay_{pp}(G,H))} deg(u)deg(v) \\ &= (deg(e)\,deg(x)) + (deg(e)\,deg(x^2)) + (deg(e)\,deg(x^3)) \\ &+ (deg(x)\,deg(x^2)) + (deg(x)\,deg(x^3)) + (deg(x^2)\,deg(x^3)) \\ &= 54. \end{aligned}$$

Meanwhile, by using 0,

$$\begin{aligned} W(UCay_{pp}(G,H)) &= \frac{2^2(2^2-1)}{2} = 6, \\ \sigma(UCay_{pp}(G,H)) &= 1, \\ M_1(UCay_{pp}(G,H)) &= 2^2(2^2-1)^2 = 36, \\ M_2(UCay_{pp}(G,H)) &= \frac{2^2(2^2-1)^3}{2} = 54. \end{aligned}$$

In the following sub-section, we present results on the general structure of the union prime power order Cayley graph for cyclic groups of order p^3, where p is a prime, as well as the topological indices of this graph.

3.3 The Union Prime Power Order Cayley Graph of Cyclic Groups of Order p^3

Let G be a cyclic group of order p^3, for prime p. For any element $x^r \in G$, the order of x^r can be $1, p, p^2$ or p^3. By 0, there are three subsets associated to G, which are:

i) $S^{(p)} = \{x^r \in G: |x^r| = p\}$,
ii) $S^{(p^2)} = \{x^r \in G: |x^r| = p^2\}$,
iii) $S^{(p^3)} = \{x^r \in G: |x^r| = p^3\}$.

Thus, the possible unions of subsets $S^{(p)}$, $S^{(p^2)}$ and $S^{(p^3)}$ can be defined as follows:

i) $H_1 = S^{(p)} \cup S^{(p^2)} = \{x^r \in G: |x^r| = p \text{ or } p^2\}$,
ii) $H_2 = S^{(p)} \cup S^{(p^3)} = \{x^r \in G: |x^r| = p \text{ or } p^3\}$,
iii) $H_3 = S^{(p^2)} \cup S^{(p^3)} = \{x^r \in G: |x^r| = p^2 \text{ or } p^3\}$,
iv) $H_4 = S^{(p)} \cup S^{(p^2)} \cup S^{(p^3)} = \{x^r \in G: |x^r| = p, p^2 \text{ or } p^3\}$.

The union prime power order Cayley graph of G with respect to H_1, H_2, H_3, and H_4 are constructed as in the following theorems.

Theorem 5

Let G be a cyclic group of order p^3 generated by x, where p is a prime. Let H_1 be a subset of G defined as $H_1 = S^{(p)} \cup S^{(p^2)} = \{x^r \in G: |x^r| = p \text{ or } p^2\}$. The union prime power order Cayley graph of G with respect to H_1 is the union of p disjoint copies of complete graphs of order p^2, that is $UCay_{pp}(G, H_1) = pK_{p^2}$.

Proof. Let G be a cyclic group of order p^3 generated by x, where

$$G = \{e, x, x^2, \dots, x^p, x^{p+1}, \dots, x^{p^2}, x^{p^2+1}, \dots, x^{p^3-1}\},$$

and let $H_1 = S^{(p)} \cup S^{(p^2)} = \{x^r \in G: |x^r| = p \text{ or } p^2\}$.

Since p is a prime, the elements of G either have order 1, p, p^2 or p^3 only. By 0, two distinct vertices g and h are adjacent if $gh^{-1} \in H_1$, which implies $|gh^{-1}| = p$ or p^2. Thus, if $g \nsim h$, then $|gh^{-1}| = 1$ or p^3. If $|gh^{-1}| = 1$, then $gh^{-1} = e$, which implies $g = h$, a contradiction since g and h are distinct. Meanwhile, if $|gh^{-1}| = p^3$, then $gh^{-1} \notin H_1$. The union prime power order Cayley graph of G with respect to H_1 has vertex set $V(UCay_{pp}(G, H_1)) = G$, which can be partitioned into p subsets, defined as $V_k = \{x^{ip+(k-1)}: i \in \mathbb{N}, 0 \leq i \leq p^2 - 1\}$, where $1 \leq k \leq p$.

Case 1: Consider $g, h \in V_k$ such that $g = x^{ip+(k-1)}$ and $h = x^{jp+(k-1)}$ for $i \neq j$. Since $gh^{-1} \in H_1$, two distinct vertices in the same partition are adjacent to each other.

Case 2: Consider $g \in V_k$ and $h \in V_l$ where $k \neq l$. Since $gh^{-1} \notin H_1$, all pair of vertices in distinct partitions V_k and V_l are not adjacent.

Since the distinct partitions are disjoint, any two vertices in the same partition are adjacent, forming a complete graph with p^2 vertices, K_{p^2}. Meanwhile, any two vertices from distinct partition are not adjacent. Therefore, $UCay_{pp}(G, H_1) = \underbrace{K_{p^2} \cup \dots \cup K_{p^2}}_{p\ times} = pK_{p^2}$.

Example 4

From 0, for a cyclic group $G = C_8$ and subset $H_1 = S^{(2)} \cup S^{(4)} = \{x^r \in G: |x^r| = 2 \text{ or } 4\} = \{x^2, x^4, x^6\}$, by 0, the vertices of $UCay_{pp}(G, H_1)$ can be partitioned into two subsets, which are:

i) $V_1 = \{x^{2i}: 0 \leq i \leq 3\} = \{e, x^2, x^4, x^6\}$,
ii) $V_2 = \{x^{2i+1}: 0 \leq i \leq 3\} = \{x, x^3, x^5, x^7\}$.

The vertices in V_1 and V_2 are adjacent to each other within the respective vertex sets, but not adjacent between the partitions. For example, the vertices $e \in V_1$ and $x \in V_2$ are not adjacent since $e \cdot x^{-1} = e \cdot x^7 = x^7 \notin H_1$. From Figure 1, it can be seen that $UCay_{pp}(G, H_1)$ is the union of two disjoint copies of complete graphs of order 4, that is $UCay_{pp}(G, H_1) = 2K_4$.

Theorem 6

Let G be a cyclic group of order p^3 generated by x, where p is a prime. Let H_2 be a subset of G defined as $H_2 = S^{(p)} \cup S^{(p^3)} = \{x^r \in G: |x^r| = p \text{ or } p^3\}$. The union prime power order Cayley graph of G with respect to H_2, $UCay_{pp}(G, H_2)$ is a $(p^3 - p^2 + p - 1)$-regular graph.

Proof. Let G be a cyclic group of order p^3 generated by x, and let $H_2 = S^{(p)} \cup S^{(p^3)} = \{x^r \in G: |x^r| = p \text{ or } p^3\}$.

Since p is a prime, the elements of G either have order 1, p, p^2 or p^3 only. By 0, two distinct vertices g and h are adjacent if $gh^{-1} \in H_2$ which implies $|gh^{-1}| = p$ or p^3. Thus, if $g \nsim h$, then $|gh^{-1}| = 1$ or p^2. If $|gh^{-1}| = 1$, then $gh^{-1} = e$, which implies $g = h$, a contradiction since g and h are distinct. Meanwhile, if $|gh^{-1}| = p^2$, then $gh^{-1} \notin H_2$.

The union prime power order Cayley graph of G with respect to H_2 has vertex set $V(UCay_{pp}(G, H_2)) = G$. Two distinct vertices x^i and x^j in $V(UCay_{pp}(G, H_2))$ are adjacent only if $x^i \cdot x^{-j} \in H_2$, for $i \neq j, 0 \leq i, j \leq p^3 - 1$. Hence, each vertex x^i is adjacent to $p^3 - p^2 + p - 1$ other vertices. Therefore, $UCay_{pp}(G, H_2)$ is a $(p^3 - p^2 + p - 1)$-regular graph.

By using similar approach as in the proof of 0 and 0, the following two theorems are obtained, respectively.

Theorem 7

Let G be a cyclic group of order p^3 generated by x, where p is a prime. Let H_3 be a subset of G defined as $H_3 = S^{(p^2)} \cup S^{(p^3)} = \{x^r \in G: |x^r| = p^2 \text{ or } p^3\}$. The union prime power order Cayley graph of G with respect to H_3, $UCay_{pp}(G, H_3)$ is a $(p^3 - p)$-regular graph.

Theorem 8

Let G be a cyclic group of order p^3 generated by x, where p is a prime. Let H_4 be a subset of G defined as $H_4 = S^{(p)} \cup S^{(p^2)} \cup S^{(p^3)} = \{x^r \in G: |x^r| = p, p^2 \text{ or } p^3\}$. The union prime power order Cayley graph of G with respect to H_4 is a complete graph of order p^3, that is $UCay_{pp}(G, H_4) = K_{p^3}$.

3.4 The Topological Indices of the Union Prime Power Order Cayley Graph of Cyclic Groups of Order p^3

In Section 3.3 the union prime power order Cayley graphs of a cyclic group G of order p^3, where p is a prime, are constructed with respect to the following subsets:

i) $H_1 = S^{(p)} \cup S^{(p^2)} = \{x^r \in G: |x^r| = p \text{ or } p^2\}$,
ii) $H_2 = S^{(p)} \cup S^{(p^3)} = \{x^r \in G: |x^r| = p \text{ or } p^3\}$,
iii) $H_3 = S^{(p^2)} \cup S^{(p^3)} = \{x^r \in G: |x^r| = p^2 \text{ or } p^3\}$,
iv) $H_4 = S^{(p)} \cup S^{(p^2)} \cup S^{(p^3)} = \{x^r \in G: |x^r| = p, p^2 \text{ or } p^3\}$.

In the following theorem, the Wiener index, first Zagreb index, and second Zagreb index of the union prime power order Cayley graphs of a cyclic group G of order p^3, for prime p, with respect to the subset H_1, denoted by $W(UCay_{pp}(G, H_1))$, $M_1(UCay_{pp}(G, H_1))$ and $M_2(UCay_{pp}(G, H_1))$ are determined. The mean distance of $UCay_{pp}(G, H_1)$ is omitted since the graph is disconnected, as shown in 0.

Theorem 9

Let G be a cyclic group of order p^3 and $H_1 = S^{(p)} \cup S^{(p^2)} = \{x^r \in G: |x^r| = p \text{ or } p^2\}$. Then,

i) $W(UCay_{pp}(G, H_1)) = \frac{p^3(p^2-1)}{2}$,
ii) $M_1(UCay_{pp}(G, H_1)) = p^3(p^2 - 1)^2$,
iii) $M_2(UCay_{pp}(G, H_1)) = \frac{p^3(p^2-1)^3}{2}$.

Proof. From 0, $UCay_{pp}(G,H_1) = pK_{p^2}$. Let $n = \left|V(UCay_{pp}(G,H_1))\right|$.
By using 0 and 0,

$$W(UCay_{pp}(G,H_1)) = \underbrace{W(K_{p^2}) + W(K_{p^2}) + \cdots + W(K_{p^2})}_{p\ times}$$
$$= \sum_{i=1}^{p} W(K_{p^2}) = \frac{p^3(p^2-1)}{2}.$$

Next, by applying 0 and 0,

$$M_1(UCay_{pp}(G,H_1)) = \underbrace{M_1(K_{p^2}) + M_1(K_{p^2}) + \cdots + M_1(K_{p^2})}_{p\ times}$$
$$= \sum_{i=1}^{p} M_1(K_{p^2}) = p^3(p^2-1)^2.$$

Then, by using 0 and 0,

$$M_2(UCay_{pp}(G,H_1)) = \underbrace{M_2(K_{p^2}) + M_2(K_{p^2}) + \cdots + M_2(K_{p^2})}_{p\ times}$$
$$= \sum_{i=1}^{p} M_2(K_{p^2}) = \frac{p^3(p^2-1)^3}{2}.$$

Example 5
From 0, $UCay_{pp}(G,H_1) = 2K_4$. Hence, by 0 and 0,

$$W(UCay_{pp}(G,H_1)) = W(K_4) + W(K_4)$$
$$= \frac{4(4-1)}{2} + \frac{4(4-1)}{2} = 12.$$

Next, by 0 and 0,

$$M_1(UCay_{pp}(G,H_1)) = M_1(K_4) + M_1(K_4)$$
$$= 4(4-1)^2 + 4(4-1)^2 = 72.$$

Then, by 0 and 0,

$$M_2(UCay_{pp}(G,H_1)) = M_2(K_4) + M_2(K_4)$$
$$= \frac{4(4-1)^3}{2} + \frac{4(4-1)^3}{2} = 108.$$

Meanwhile, by using 0,

$$W(UCay_{pp}(G,H_1)) = \frac{2^3(2^2-1)}{2} = 12,$$

$M_1(UCay_{pp}(G,H_1)) = 2^3(2^2-1)^2 = 72,$
$M_2(UCay_{pp}(G,H_1)) = \dfrac{2^3(2^2-1)^3}{2} = 108.$

Next, the Wiener index, mean distance, first Zagreb index, and second Zagreb index of the union prime power order Cayley graph of a cyclic group G of order p^3, for prime p, with respect to the subsets H_2, H_3, and H_4 are determined.

First, the diameter of $UCay_{pp}(G,H_2)$ is determined as stated in the following lemma, which is necessary for computing the Wiener index of $UCay_{pp}(G,H_2)$ using 0.

Lemma 1

Let G be a cyclic group of order p^3 and $H_2 = S^{(p)} \cup S^{(p^3)} = \{x^r \in G: |x^r| = p \text{ or } p^3\}$. The diameter of $UCay_{pp}(G,H_2)$ is 2.

Proof. From 0, $UCay_{pp}(G,H_2)$ is a (p^3-p^2+p-1)-regular graph.

Case 1: Consider two distinct vertices x^i and x^j in $V(UCay_{pp}(G,H_2))$ are adjacent. If vertices x^i and x^j are adjacent, then the distance between them is one.

Case 2: Consider two distinct vertices x^i and x^j in $V(UCay_{pp}(G,H_2))$ are not adjacent. Since the graph is regular, both vertices x^i and x^j share a common neighbor. There exists a vertex x^k such that $x^k \sim x^i$ and $x^k \sim x^j$ in which x^k is adjacent to both x^i and x^j. Thus, the distance between x^i and x^j is two.

The distance between any two vertices in $UCay_{pp}(G,H_2)$ is at most 2. Therefore, the diameter of $UCay_{pp}(G,H_2)$ is 2.

Theorem 10

Let G be a cyclic group of order p^3 and $H_2 = S^{(p)} \cup S^{(p^3)} = \{x^r \in G: |x^r| = p \text{ or } p^3\}$. Then,

i) $W(UCay_{pp}(G,H_2)) = \frac{p^3(p^3+p^2-p-1)}{2}$,
ii) $\sigma(UCay_{pp}(G,H_2)) = \frac{(p+1)^2}{p^2+p+1}$,
iii) $M_1(UCay_{pp}(G,H_2)) = p^3(p^3-p^2+p-1)^2$,
iv) $M_2(UCay_{pp}(G,H_2)) = \frac{(p^4-p^3+p^2-p)^3}{2}$.

Proof. From 0, $UCay_{pp}(G,H_2)$ is a (p^3-p^2+p-1)-regular graph. Based on 0, $diam(UCay_{pp}(G,H_2)) = 2$. Let $n = |V(UCay_{pp}(G,H_2))|$ and $d = p^3-p^2+p-1$.

The number of edges in a d-regular graph with n vertices is $\frac{nd}{2}$. Hence, by 0,

$$
\begin{aligned}
W(UCay_{pp}(G,H_2)) &= |V(UCay_{pp}(G,H_2))|^2 - |V(UCay_{pp}(G,H_2))| - |E(UCay_{pp}(G,H_2))| \\
&= (p^3)^2 - p^3 - \frac{p^3(p^3-p^2+p-1)}{2} \\
&= \frac{p^3(p^3+p^2-p-1)}{2}.
\end{aligned}
$$

Next, by 0, the mean distance of $UCay_{pp}(G,H_2)$ is,

$$\sigma(UCay_{pp}(G,H_2)) = \frac{W(UCay_{pp}(G,H_2))}{\binom{|V(UCay_{pp}(G,H_2))|}{2}} = \frac{\frac{p^3(p^3+p^2-p-1)}{2}}{\binom{p^3}{2}} = \frac{(p+1)^2}{p^2+p+1}.$$

Then, by using 0 and 0, the first and second Zagreb indices of $UCay_{pp}(G,H_2)$ are determined, respectively.

$$M_1(UCay_{pp}(G,H_2)) = p^3(p^3-p^2+p-1)^2,$$
$$M_2(UCay_{pp}(G,H_2)) = \frac{p^3(p^3-p^2+p-1)^3}{2} = \frac{(p^4-p^3+p^2-p)^3}{2}.$$

By using similar approach as in the proof of 0 and 0, the following two theorems are obtained, respectively.

Theorem 11
Let G be a cyclic group of order p^3 and $H_3 = S^{(p^2)} \cup S^{(p^3)} = \{x^r \in G: |x^r| = p^2 \text{ or } p^3\}$. Then,

i) $W(UCay_{pp}(G,H_3)) = \frac{p^3(p^3+p-2)}{2}$,
ii) $\sigma(UCay_{pp}(G,H_3)) = \frac{p^3+p-2}{p^3-1}$,
iii) $M_1(UCay_{pp}(G,H_3)) = p^5(p^2-1)^2$,
iv) $M_2(UCay_{pp}(G,H_3)) = \frac{p^6(p^2-1)^3}{2}$.

Theorem 12
Let G be a cyclic group of order p^3 and $H_4 = S^{(p)} \cup S^{(p^2)} \cup S^{(p^3)} = \{x^r \in G: |x^r| = p, p^2 \text{ or } p^3\}$. Then,

i) $W(UCay_{pp}(G,H_4)) = \frac{p^3(p^3-1)}{2}$,
ii) $\sigma(UCay_{pp}(G,H_4)) = 1$,
iii) $M_1(UCay_{pp}(G,H_4)) = p^3(p^3-1)^2$,
iv) $M_2(UCay_{pp}(G,H_4)) = \frac{p^3(p^3-1)^3}{2}$.

4. Conclusions

In this paper, a new variant of Cayley graph, called the union prime power order Cayley graph is introduced. This graph is constructed for a cyclic group G of order p^2, where p is a prime, and is found to be a complete graph with p^2 vertices, K_{p^2}. Based on 0 and 0, it can be observed that $UCay_{pp}(G,H) \cong \widetilde{Cay}(G,S)$ for a cyclic group G of order p^2. Additionally, general formulas for the topological indices of this graph are established using existing results on topological indices of complete graphs.

Furthermore, for a cyclic group G of order p^3, the structures of the union prime power order Cayley graphs with specific subsets H_1, H_2, H_3, and H_4 , and their topological indices, are generalized. Similarly, it is observed that the union prime power order Cayley graph of G with respect to H_4, $UCay_{pp}(G, H_4)$ is isomorphic to $\widetilde{Cay}(G, S)$ for a cyclic group G of order p^3.

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