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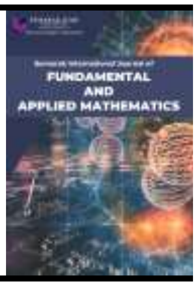

Application of the Banach Contraction Method to the Benjamin-Bona Mahony Equation: Comparative Analysis with the Laplace Adomian Decomposition Method

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ABSTRACT

The Benjamin-Bona Mahony (BBM) equation, a nonlinear dispersive wave model, plays a crucial role in fields such as fluid dynamics and plasma physics. Solving the BBM equation analytically is challenging, necessitating numerical and semi-analytical approaches. This study investigates the application of the Banach Contraction Method (BCM) to the BBM equation, comparing its performance with the Laplace Adomian Decomposition Method (LADM). By employing iterative approximations, BCM demonstrates convergence to unique solutions under specific conditions, ensuring reliability. Two examples of the BBM equation are analyzed, and absolute differences between BCM and LADM results are evaluated for varying spatial and temporal resolutions. The results reveal that both methods exhibit high accuracy, with smaller discrepancies observed for shorter time intervals. However, differences increase with spatial position, suggesting potential sensitivity to spatial dynamics. BCM shows an advantage over LADM due to its strong theoretical framework for ensuring convergence and uniqueness. While LADM offers flexibility in nonlinear term handling, BCM's robustness makes it preferable in critical applications. Recommendations for further research include exploring computational efficiency, extending the comparison to other wave equations, and analyzing higher-order solutions to broaden the applicability of these method.

1. Introduction

The Benjamin-Bona Mahony (BBM) equation, introduced as an enhancement of the Korteweg-de Vries (KdV) equation, is a fundamental model for describing nonlinear and dispersive wave propagation. Its applications span fluid dynamics, plasma physics, and meteorology, where it effectively captures the behaviour of long waves in nonlinear dispersive. By incorporating both nonlinearity and dispersion effects, the BBM equation has become a pivotal tool for understanding wave dynamics in various scientific and engineering domains.

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Solving the BBM equation analytically is often challenging due to its inherent complexity. Consequently, numerical and semi-analytical methods have become essential tools. Among these methods, the Banach Contraction Method (BCM) has garnered attention for its ability to provide reliable solutions for nonlinear differential equations. By leveraging fixed-point theory, BCM ensures convergence to a unique solution under specific conditions. This article explores the application of BCM to solve the BBM equation, focusing on its reliability, computational efficiency, and potential for broader applications.

Over the years, researchers have proposed various methods to address the BBM equation, emphasizing both analytical and numerical approaches. These methods have significantly advanced the understanding and solution of this important equation.

Analytical solutions provide valuable insights into the behavior of wave phenomena described by the BBM equation. Abbasbandy *et al.,* [1] utilized the first integral method to derive new exact solutions for the modified BBM equation, revealing intricate relationships between wave parameters. Noor *et al.,* [2] introduced the exp-function method, showcasing its capability to compute soliton solutions efficiently. Yokus *et al.,* [3] explored the sine-Gordon expansion method, deriving solutions involving hyperbolic functions, which are crucial for understanding wave structures. Despite their utility, these approaches often rely on simplifying assumptions, limiting their applicability to more generalized scenarios. To overcome these limitations, researchers have turned to numerical techniques.

Numerical methods are indispensable for solving the BBM equation in complex scenarios. Shi *et al.,* [4] developed a Crank-Nicolson finite element method, emphasizing energy conservation and unconditional super convergence. Zhou *et al.,* [5] proposed a predictor-corrector method tailored for time-fractional BBM equations, demonstrating increased accuracy and reduced computational cost. Similarly, Li *et al.,* [6] employed local discontinuous Galerkin methods to achieve discrete conservation of mass and energy.

Wavelet-based numerical techniques have also gained prominence. Shiralashetti *et al.,* [7] applied Taylor wavelets in a collocation framework to solve BBM equations, achieving uniform convergence and reliable solutions. Mulimani *et al.,* [8] proposed an ultraspherical wavelet method, showcasing its speed, flexibility, and convergence.

The fractional versions of the BBM equation, modeling nonlocal and nonlinear phenomena, have been extensively studied. Oruc *et al.,* [9] explored fractional BBM equations, deriving solitary wave solutions validated through numerical experiments. Abbas *et al.,* [10] extended this analysis to the Wazwaz BBM equation, using optimal Lie infinitesimal generators to obtain traveling wave profiles.

Modifications of the BBM equation often introduce additional complexities. Devi *et al.,* [11] employed higher-order shape elements in the Galerkin finite element method to solve BBM-Burgers equations, achieving improved accuracy. Such studies underline the importance of innovative methods in tackling advanced forms of the BBM equation.

Iterative methods have become popular for solving nonlinear differential equations due to their adaptability and computational efficiency. Olubanwo *et al.,* [12] combined the Laplace transform with homotopy perturbation to address nonlinear terms in the BBM equation.

Hybrid approaches incorporating numerical and analytical techniques have also shown promise. For example, Raslan *et al.,* [13] integrated BCM with other methods to solve the Drinfeld-Sokolov-Wilson system, demonstrating reduced errors and enhanced efficiency.

BCM, rooted in fixed-point theory, offers a robust framework for solving nonlinear equations. By iteratively applying a contraction mapping, the method ensures convergence to a unique fixed point. Kittisopaporn *et al.,* [14] employed BCM for Sylvester matrix equations, validating its convergence and reliability. Ghitheeth *et al.,* [15] demonstrated the method's efficiency in nonlinear functional equations when combined with numerical techniques like the trapezoid rule.

The application of BCM to the BBM equation remains a relatively unexplored area. However, its proven effectiveness in other nonlinear and dispersive systems highlights its potential as a powerful tool for solving the BBM equation. In addition to introducing BCM to the BBM equation, this study provides a comparative analysis of BCM with the Laplace Adomian Decomposition Method (LADM), highlighting methodological variances and their implications for solution accuracy. This comparison offers deeper insights into the strengths and limitations of these methods, especially in regions with steep gradients or rapidly varying initial conditions. Furthermore, this work emphasizes the BBM equation's practical significance in real-world applications, such as modeling tidal waves and plasma instabilities in fusion reactors, where accurate and efficient computational solutions are critical.

2. Methodology

2.1 Banach Contraction Method

We start this section by stating some basic concepts [16].

Definition: *Let X_1 and X_2 be two metrics and F be a mapping from X_1 into X_2. F is said to be Lipschitz if there exists a real number $r \geq 0$ for all $x_1, x_2 \in X$we have $d(Fx_1, Fx_2) \leq rd(x_1, x_2)$, F is said to contraction mapping if $r < 1$.*

Theorem 1 *Let F be contraction mapping with a Lipschitz constant r, of a complete metric space X into itself, then F has a unique fixed point u in the space X. An addition, if x_0 is an arbitrary point in X and x is defined by $x_{n+1} = F(x_n)$, $n = 0,1,2\ldots$, the $\lim_{x\to\infty} x_n = u$ and $d(x_n, u) \leq \frac{r^n}{1-r} d(x_1, x_0)$.*

Theorem 2 *Let F be a mapping of a complete metric space X into itself such that F^k is a contraction mapping of X for some positive integer k, then F has a unique fixed point in X.*

To illustrate the basic concept of BCM, we introduce the following general form of differential equation:

$$D_t^n u(t) = L(u(t)) + K(u(t)) + g(t), n \in \mathbb{N}, \tag{1}$$

with initial condition

$$\frac{d^k}{dt^k} u(0) = h_k, \quad k = 0,1,2,\ldots,n-1, \tag{2}$$

where L, K are linear and nonlinear operators of orders less than or equal to n, $g(t)$ is a non-homogeneous term and D_n^t classical differential operator of order n. Applying the classical integral operator with n fold with respect to t, denoted by I_n^t, to both sides Eq. (1) we can obtain the following integral equation:

$$u(t) = \sum_{k=0}^{n-1} h_k \frac{t^k}{k!} + I_n^t g(t) + I_n^t (L(u) + K(u)). \tag{3}$$

To implement the BCM, we consider Eq. (3) as a general functional equation

$$u(t) = f + N(u(t)), \tag{4}$$

where $f = \sum_{k=0}^{n-1} h_k \frac{t^k}{k!} + I_t^n g(t)$ and $N(u) = I_n^t(L(u) + K(u))$.

Then we define the successive approximation [17],

$$
\begin{aligned}
&u_0(t) = f,\\
&u_1(t) = u_0(t) + N(u_0(t)),\\
&u_2(t) = u_0(t) + N(u_1(t)),\\
&\vdots\\
&u_{n+1}(t) = u_0(t) + N(u_n(t)), \quad n = 1,2, \ldots .
\end{aligned}
\tag{5}
$$

If N^k is the contraction for some positive integer, $k$then $N(u(t))$ has a unique fixed point. Hence the sequence defined by Eq. (5) is convergent according to Theorem 2 and the solution of Eq. (1) is given by

$$u(t) = \lim_{n\to\infty} u_n(t). \tag{6}$$

2.2 The Benjamin-Bona-Mahonay Equations

The BBM equation, a partial differential equation, models the propagation of long waves in nonlinear dispersive media. It is expressed as:

$$u_t + \alpha u_x + u^n u_x - u_{xxt} = 0 \tag{7}$$

In this equation, u represents the wave function, which depends on both time, t and the spatial coordinate x. The constants α and n are arbitrary. The term u_t denotes the rate of change of u with respect to time, while u_x signifies its spatial gradient. The nonlinear term, $u^n u_x$, reflects the interaction between u and its spatial gradient, introducing nonlinearity into the equation.

2.3 Implementation of BCM on The Benjamin-Bona-Mahonay Equations

For this purpose, we considered the BBM Eq. (7) with $\alpha = 1$ and $n = 1$ The initial condition is given by $u(x, 0) = g(x)$. Because of the BCM, we obtained:

$$
\begin{aligned}
&u_0 = \int g(x)dt,\\
&u_1 = u_0 - \int (u_{0x} + u_0 u_{0x} - u_{0xxt})dt,\\
&u_2 = u_0 - \int (u_{1x} + u_1 u_{1x} - u_{1xxt})dt,\\
&\vdots\\
&u_{n+1} = u_0 - \int (u_{nx} + u_n u_{nx} - u_{nxxt})dt.
\end{aligned}
\tag{8}
$$

Therefore, the solution yield as:

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{9}$$

3. Results and Discussion

3.1 Example 1

We considered the BBM Eq. (7) where $\alpha = 1$ and $n = 1$ and $g(x) = e^x$. Using the BCM we obtained the following iterative solutions:

$$u_0(x,t) = e^x,$$
$$u_1(x,t) = e^x - (e^x + e^{2x})t,$$
$$u_2(x,t) = e^x - e^x\left(\frac{2e^{3x}t^3}{3} + e^{2x}t^3 - \frac{3e^{2x}t^2}{2} + \frac{e^x t^3}{3} - 2e^x t^2 + 5e^x t - \frac{t^2}{2} + 2t\right), \quad (10)$$
$$\vdots$$

and so on.

Figures 1- 3 illustrate the solution of the BBM equation $u_2(x,t)$, using LADM [18] and BCM $t = 0.001$, $t = 0.01$ and $t = 0.1$, respectively. At $t = 0.001$ both methods produce almost identical results, showing high accuracy over small time intervals when nonlinear effects and dispersion are minimal. At $t = 0.01$, a slight difference in values starts to appear, especially when moving away from the spatial position x with the BCM giving a steeper decline of $u_2(x,t)$, compared to the smoother behavior of LADM. For $t = 0.1$, significant differences between the two methods are evident, especially for larger x. While the BCM shows a rapid and abrupt decrease, characteristic of nonlinear effects and strong dispersion, the LADM produces an increasing trend in $u_2(x,t)$, for the same region, deviating from the expected behavior of the solution. This shows that the BCM provides a more stable and accurate representation of the wave dynamics in this scenario, while the LADM struggles with capturing the qualitative behavior of the solution as x increases, possibly due to resistance in handling non-linearity or fundamental dispersion at this spatial region.

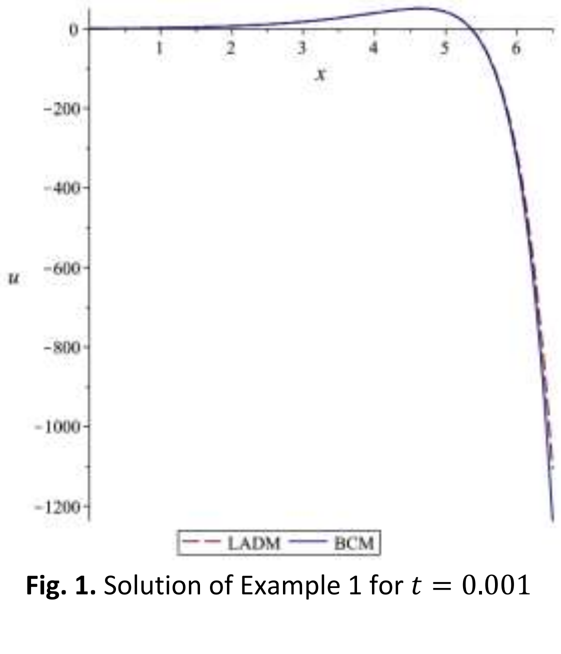

Fig. 1. Solution of Example 1 for $t = 0.001$

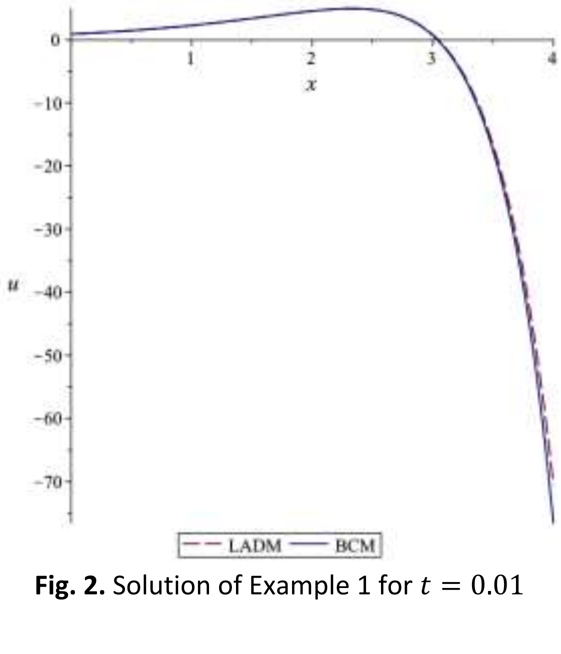

Fig. 2. Solution of Example 1 for $t = 0.01$

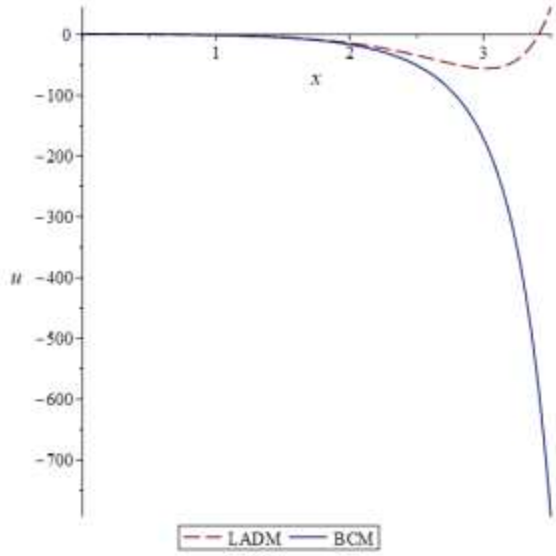

Fig. 3. Solution of Example 1 for $t = 0.1$

3.2 Example 2

We considered the BBM Eq. (7) where $\alpha = 1$ and $n = 1$ and $g(x) = x^2$. Using the BCM we obtained the following iterative solutions:

$$u_0(x,t) = x^2,$$
$$u_1(x,t) = x^2 - (2x^3 + 2x)t,$$
$$u_2(x,t) = x^2 - 14xt + \frac{(6x^2+2)t^2}{2} - \frac{(-2x^3-2x)(-6x^2-2)t^3}{3}$$
$$- \frac{(x^2(-6x^2-2)+2(-2x^3-2x)x)t^2}{2} - 2x^3 \quad (11)$$
$\vdots$
and so on.

Figures 4-6 display the solution of the BBM equation using LADM [18] and BCM for $t = 0.001$, $t = 0.01$ and $t = 0.1$, respectively. At $t = 0.001$, both methods' results are almost perfectly aligned, reflecting their reliability for small time intervals when nonlinear effects and dispersion are still minimal. At $t = 0.01$, both methods exhibit similar behaviour, with $u_2(x,t)$ increasing monotonically as x increase. The results closely aligned across most domains, indicating that both methods capture the solution's overall trend well. However, minor discrepancies emerge at larger x, where LADM slightly overestimates the solution compared to BCM. For $t = 0.1$ initially, both methods produce the same result, with the solution $u_2(x,t)$ starting at zero. However, when x increases, significant differences occur. The LADM solution grows faster than the BCM solution, leading to an overestimation of $u_2(x,t)$ for higher x distances. BCM shows better accuracy and stability for modeling the BBM equation, especially when x becomes larger.

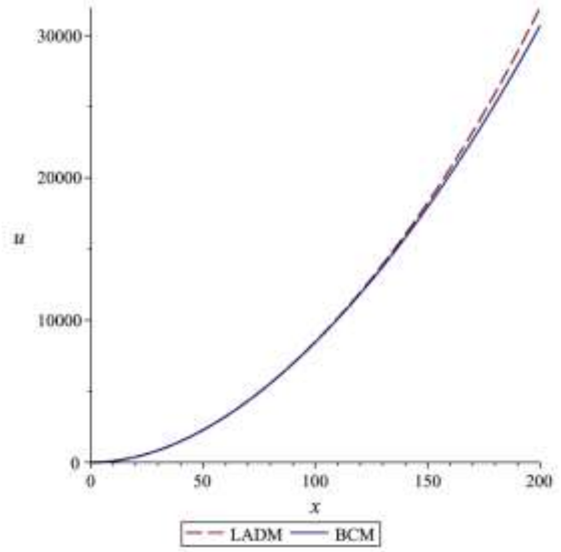

Fig. 4. Solution of Example 1 for $t = 0.001$

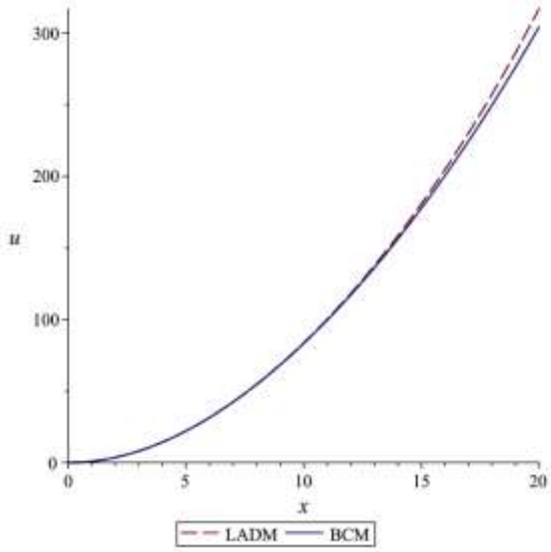

Fig. 5. Solution of Example 2 for $t = 0.01$

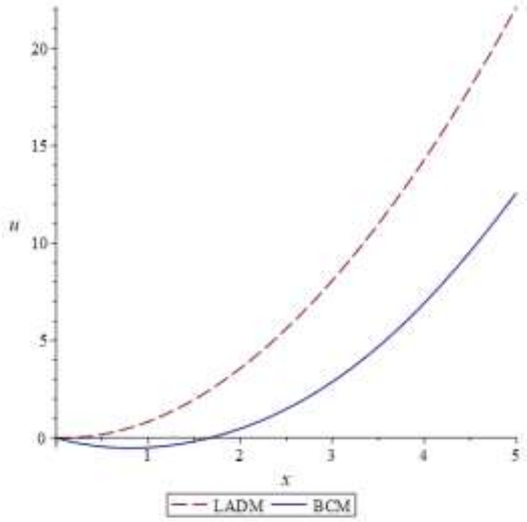

Fig. 6. Solution of Example 2 for $t = 0.1$

3.3 Error Analysis

To determine the efficiency of BCM, we compared the results with the results by Ikram et al., [18] using the Laplace Adomian decomposition method (LADM) for spatial domain $x \in [0,1]$. This short distance was chosen because the solutions from both methods are expressed as power series, which converge only within a limited range around the starting point.

Table 1 compares the absolute differences between results obtained using the BCM and LADM for $u_2(x,t)$, a solution to the BBM equation Example 1, across different spatial positions x and time t. The results show that as t decreases ($0.1 \rightarrow 0.01 \rightarrow 0.001$), the differences become significantly smaller, indicating closer agreement between the methods for shorter time intervals. Conversely, for a fixed t, the differences increase with x, suggesting slight divergence between the methods as the spatial position moves further from the origin.

Overall, the small magnitudes of the differences (on the order of 10^{-3} to 10^{-9} demonstrate that BCM and LADM are highly consistent and accurate for solving the BBM equation. The results highlight both methods' reliability, especially for smaller t, while the growth in discrepancies with x points to potential numerical or methodological variances that may require further examination.

Table 1
The absolute difference between results by BCM, $u_2(x,t)$ and LADM, $u_2(x,t)$ [18]

x	t = 0.1	t = 0.01	t = 0.001
0	2.000E-03	2.000E-06	2.000E-09
0.1	2.752E-03	2.752E-06	3.000E-09
0.2	3.803E-03	3.803E-06	4.000E-09
0.3	5.280E-03	5.280E-06	5.000E-09
0.4	7.364E-03	7.364E-06	7.000E-09
0.5	1.031E-02	1.031E-05	1.000E-08
0.6	1.451E-02	1.450E-05	1.500E-08
0.7	2.048E-02	2.048E-05	2.000E-08
0.8	2.903E-02	2.903E-05	2.900E-08
0.9	4.130E-02	4.129E-05	4.100E-08
1	5.895E-02	5.895E-05	5.900E-08

Table 2 presents the absolute differences between the results of the BCM and LADM for $u_2(x,t)$, a solution to the BBM equation Example 2, evaluated over different spatial positions x and times t. At $t = 0.1$, the differences grow with x, starting from zero at $x = 0$ and increasing up to 0.01067 at $x = 1$. For smaller t, the differences decrease significantly, showing values on the order of 10^{-7} for $t = 0.01$ and 10^{-10} for $t = 0.001$, indicating closer agreement between the two methods as time decreases.

The results highlight that BCM and LADM are highly consistent, particularly for smaller t, where the differences are negligible. The discrepancies increase with x for all times, suggesting minor variations in how the methods handle spatial dynamics. This reinforces the reliability of both methods while indicating that the differences may become more pronounced at larger spatial positions, especially for longer time intervals

Table 2
Absolute difference between results by BCM, $u_2(x,t)$ and LADM, $u_2(x,t)$ [18]

x	t = 0.1	t = 0.01	t = 0.001
0	0.000E+00	0.000E+00	0.000E+00
0.1	1.387E-04	1.387E-07	1.390E-10
0.2	3.106E-04	3.106E-07	3.100E-10
0.3	5.537E-04	5.537E-07	5.500E-10
0.4	9.156E-04	9.156E-07	9.000E-10
0.5	1.458E-03	1.458E-06	1.500E-09
0.6	2.263E-03	2.263E-06	2.300E-09
0.7	3.435E-03	3.435E-06	3.400E-09
0.8	5.108E-03	5.108E-06	5.100E-09
0.9	7.450E-03	7.450E-06	7.400E-09
1	1.067E-02	1.067E-05	1.070E-08

4. Conclusion and Recommendation

The comparative analysis of Figures 1-6 demonstrates the performance of the Laplace Adomian Decomposition Method (LADM) and the Banach Contraction Method (BCM) in solving the BBM equation across different time and spatial domains. For smaller time ($t = 0.01,\ t = 0.001$), both methods exhibit close agreement, affirming their accuracy and convergence in modeling the BBM equation dynamics. However, as the spatial domain or time increases, discrepancies become more pronounced, with LADM consistently overestimating the solution compared to BCM. This pattern suggests that LADM is well-suited for localized or short-range computations but may accumulate errors over larger domains. In contrast, BCM provides smoother and more stable solutions, showcasing its superior numerical stability and reliability, particularly for extended spatial computations.

The results from Tables 1 and 2 further highlight BCM's advantage in ensuring accuracy, especially as the time step decreases, with smaller absolute differences observed for shorter time intervals. While both methods perform well for small spatial variables, the growing discrepancies with increasing spatial domains underscore the importance of spatial dynamics in the solution's accuracy. BCM's strong theoretical foundation, supported by the Banach contraction principle, ensures robustness, convergence, and uniqueness, making it the preferred choice for problems requiring high reliability. However, LADM remains competitive due to its flexibility in handling nonlinear terms and its efficiency in requiring fewer iterations for convergence in certain cases.

Based on these findings, BCM is recommended for solving BBM equations in larger spatial domains or for scenarios demanding high numerical stability and precision. LADM is better suited for quick approximations and localized solutions. Future research should focus on the computational efficiency of both methods, their scalability to higher-order solutions, and their performance under varying parameter settings. Extending the comparative analysis to other nonlinear wave equations could provide broader insights into the applicability and limitations of these methods.

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